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Logics for first-order team properties

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Abstract. In this paper, we introduce a logic based on team semantics, called **FOT**, whose expressive power coincides with first-order logic both on the level of sentences and (open) formulas, and we also show that a sublogic of **FOT**, called **FOT**[↓], captures exactly downward closed first-order team properties. We axiomatize completely the logic **FOT**, and also extend the known partial axiomatization of dependence logic to dependence logic enriched with the logical constants in **FOT**[↓].

Keywords: Dependence logic · Team semantics · First-order logic.

1 Introduction

In this paper, we define logics based on team semantics for characterizing first-order team properties, and we also study the axiomatization problem of these logics.

Team semantics is a semantical framework originally introduced by Hodges [21], and later systematically developed by Väänänen with the introduction of *dependence logic* [31], which extends first-order logic with *dependence atoms*. Other notable logics based on team semantics include *independence logic* introduced by Grädel and Väänänen [16] (which is first-order logic extended with independence atoms), and *inclusion logic* introduced by Galliani [11] (which is first-order logic extended with inclusion atoms). In team semantics formulas are evaluated in a model over *sets* of assignments for the free variables (called *teams*) rather than single assignments as in the usual first-order logic. Teams X with the domain $\{v_1, \dots, v_k\}$ are essentially k -ary relations $rel(X) = \{(s(v_1), \dots, s(v_k)) \mid s \in X\}$, and thus open formulas define team properties. In general, knowing the expressive power of a logic for sentences (with no free variables) does not automatically give a characterization for the expressive power of open formulas of the same logic. Such a peculiar phenomenon has sparked several studies on the expressive power of logics based on team semantics. In particular, while it follows straightforwardly from the earlier known results of Henkin, Enderton, Walkoe, and Hodges [20,8,32,22] that dependence logic (**D**) and independence logic (**Ind**) are both equivalent to existential second-order logic (**ESO**) on the level of sentences, it turns out that open formulas of **D** have different expressive power from open formulas of **Ind**: The latter characterize all **ESO** team properties [11], whereas the former characterize only downward closed **ESO** team properties [26]. Along the same line, a later breakthrough showed that inclusion logic corresponds, over sentences, to *positive greatest fixed-point logic* [15], which is strictly more expressive than first-order logic as well.

In this paper we define a team-based logic, called **FOT**, whose expressive power coincides with first-order logic (**FO**) both on the level of sentences and open formulas, in the sense that **FOT**-formulas characterize (modulo the empty team) exactly team properties definable by first-order sentences with an extra relation symbol R . To the best of the authors' knowledge, no such logic has been defined previously.

In related previous work, it was shown in [10,13,28] that first-order logic extended with *constancy atoms* $=(x)$ and **FO** extended with classical negation $\sim$ are both equivalent to **FO** over sentences, whereas on the level of formulas they are both strictly less expressive than **FO**, and thus fail to capture all first-order team properties. It was also illustrated in [24] that a certain simple disjunction of dependence atoms already defines an NP-complete team property. Therefore, any logic based on team semantics having the disjunction $\vee$ inherited from first-order logic and in which dependence atoms are expressible will be able to express NP-complete team properties indicating that $\vee$ is too expressive connective to be added to **FOT**. The logic **FOT** we define in this paper has weaker version of disjunction $\vee$ and classical negation $\sim$ as well as weaker quantifiers $\forall^1, \exists^1$. We prove, in Section 3, that our logic **FOT** captures first-order team properties (modulo the empty team) and we also show, as an application of Lyndon's Interpolation Theorem of first-order logic, that a sublogic of **FOT**, denoted as **FOT** $^\downarrow$, captures exactly downward closed first-order team properties (modulo the empty team).

In the second part of this paper study the axiomatization problem of our logics **FOT** and **FOT** $^\downarrow$. In Section 4 we introduce a sound and complete system of natural deduction for **FOT** that on one hand behaves like the system of **FO** to a certain extend (in the sense of Lemma 12), while on the other hand incorporates natural and interesting rules for inclusion atoms and their interaction with the weak logical constants.

In Section 5, we apply our results to the problem of finding axiomatizations for larger and larger fragments of dependence logic and its variants by extending the known partial axiomatization of dependence logic to **D** enriched with the logical constants in **FOT** $^\downarrow$ (denoted as **D** $\oplus$ **FOT** $^\downarrow$), which, by our result in the first part of this paper, is expressively equivalent to **D**. While **D** is not effectively axiomatizable (for it is equivalent to **ESO**), a complete axiomatization for first-order consequences of **D**-sentences has been given in [27]. More precisely, a system of natural deduction for dependence logic was introduced in [27] for which the completeness theorem

$$\Gamma \vdash \theta \iff \Gamma \models \theta \quad (1)$$

holds whenever Γ is a set of **D**-sentences and θ is an **FO**-sentence. This result has been, subsequently, generalized to e.g., allows also open formulas [25], and treats also independence logic [17] or dependence logic with generalized quantifiers [9]. A recent new generalization given in [33] extends the known systems for **D** and **Ind** to cover the case when θ in (1) is not necessarily an **FO**-formula but merely a formula defining a first-order team property. However, since the problem of whether a **D**- or **Ind**-formula defines a first-order team property is undecidable, the extension of [33] is not effectively represented. Motivated by the results of [33], we give an effective extension of (1), in which θ is a sentence of our compositionally defined logic **FOT** $^\downarrow$ and Γ is a set of **D** $\oplus$ **FOT** $^\downarrow$ -sentences. Finding an effective axiomatization in the more general case of **Ind** enriched with the logical constants in **FOT** is left as future work.

Apart from theoretical significance, our results also provide new logical tools for the applications of team-based logics in other related areas; such applications have been studied in recent years, e.g., in database theory [19], formal semantics of natural language [4,5], Bayesian statistics [18,7], social choice theory [30], and quantum information theory [23]. In particular, inquisitive logic [6] adopts, independently, also the team semantics to provide formal semantics of questions in natural language, and the first-order version of inquisitive logic can be viewed as a team-based logic (in a slightly different setting) with the weak disjunction $\vee$ and the weak quantifiers $\forall^1, \exists^1$. The study we provide in this paper for the expressive power and axiomatization problem of these weaker logical constants will potentially help clarifying properties of first-order inquisitive logic. In the recent formalization of Arrow's Theorem [2] in social choice in independence logic [30], the weak disjunction $\vee$ plays a natural role, and the completeness theorem of the type (1) was crucial for deriving Arrow's Theorem formally. The axiomatization results we obtained in this paper are then expected to contribute to the formal analysis of Arrow's Theorem and other impossibility theorems in social choice.

2 Preliminaries

We consider first-order vocabularies $\mathcal{L}$ with an equality symbol $=$. An $\mathcal{L}$ -term t is defined inductively as usual, and formulas of *First-order Logic* (**FO**) are defined as:

$$\alpha ::= t_1 = t_2 \mid R t_1 \dots t_k \mid \neg \alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x \alpha \mid \forall x \alpha.$$

Throughout the paper, we reserve the first Greek letters $\alpha, \beta, \gamma, \delta$ for first-order formulas. As usual, define $\alpha \rightarrow \beta := \neg \alpha \vee \beta$. We use the letters $v, x, y, z, \dots$ in sans-serif face to stand for sequences of variables, and sequences of terms are denoted as $t, t', \dots$. We write $\text{Fv}(\alpha)$ for the set of free variables of α , and write $\alpha(x)$ to indicate that the free variables of α are among $x = \langle x_1, \dots, x_n \rangle$. A formula with no free variables is called a *sentence*.

For any $\mathcal{L}$ -model M , we use the same notation M also to denote its domain. We write $\mathcal{L}(R)$ for the vocabulary expanded from $\mathcal{L}$ by adding a fresh relation symbol R , and write (M, R^M) for the $\mathcal{L}(R)$ -expansion of M in which the k -ary relation symbol R is interpreted as $R^M \subseteq M^k$. We sometimes write $\alpha(R)$ to emphasize that the first-order formula α is in the vocabulary $\mathcal{L}(R)$ for some $\mathcal{L}$.

We assume that the reader is familiar with the usual Tarskian semantics of first-order logic. In this paper, we consider logics with *team semantics*. A *team* X of M over a set V of variables is a set of assignments $s : V \rightarrow M$, where V is called the domain of X , denoted $\text{dom}(X)$. Given a first-order formula α , given any $\mathcal{L}$ -model M and any team X over $V \supseteq \text{Fv}(\alpha)$, we define the *satisfaction* relation $M \models_X \alpha$ inductively as follows:

- $M \models_X \lambda$ for λ a first-order atom iff for all $s \in X$, $M \models_s \lambda$ in the usual sense.
- $M \models_X \neg \alpha$ iff for all $s \in X$, $M \not\models_{\{s\}} \alpha$.
- $M \models_X \alpha \wedge \beta$ iff $M \models_X \alpha$ and $M \models_X \beta$.
- $M \models_X \alpha \vee \beta$ iff there are $Y, Z \subseteq X$ such that $X = Y \cup Z$, $M \models_Y \alpha$ and $M \models_Z \beta$.
- $M \models_X \exists x \alpha$ iff $M \models_{X(F/x)} \alpha$ for some $F : X \rightarrow \wp(M) \setminus \{\emptyset\}$, where $X(F/x) = \{s(a/x) \mid s \in X, a \in F(s)\}$.
- $M \models_X \forall x \alpha$ iff $M \models_{X(M/x)} \alpha$, where $X(M/x) = \{s(a/x) \mid s \in X, a \in M\}$.

It is easy to verify that first-order formulas have the following properties:

Empty team property: $M \models_{\emptyset} \phi$

Downward closure: $[M \models_X \phi \text{ and } Y \subseteq X] \implies M \models_Y \phi$.

Union closure: $[M \models_X \phi \text{ and } M \models_Y \phi] \implies M \models_{X \cup Y} \phi$.

Downward closure property together with union closure property are equivalent to

Flatness property: $M \models_X \phi \iff M \models_{\{s\}} \phi$ for all $s \in X$.

Logics based on team semantics do not in general have the flatness property. For instance, *dependence logic* [31], which is first-order logic extended with dependence atoms $=(t_1, \dots, t_n, t)$, is downward closed but not flat; and *inclusion logic* [11], which is first-order logic extended with inclusion atoms $t_1, \dots, t_n \subseteq t'_1, \dots, t'_n$, is union closed but not flat. Especially, dependence atoms and inclusion atoms are not flat. We recall their semantics below:

- $M \models_X =(t, t')$ iff for all $s, s' \in X$, $s(t) = s'(t)$ implies $s(t') = s'(t')$.
- $M \models_X t \subseteq t'$ iff for all $s \in X$, there is $s' \in X$ such that $s(t) = s'(t')$

In this paper, we study two (non-flat) logics based on team semantics, called **FOT** and **FOT[↓]**, whose formulas are built from a different (yet similar) set of connectives and quantifiers than those in first-order logic as follows:

(FOT) $\phi ::= \lambda \mid x \subseteq y \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists^1 x \phi \mid \forall^1 x \phi$

(FOT[↓]) $\phi ::= \lambda \mid \neg \delta \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists^1 x \phi \mid \forall^1 x \phi$

where λ is an arbitrary first-order atomic formula, x and y are two sequences of variables of the same length, and δ is a quantifier-free and disjunction-free formula (i.e., $\delta ::= \lambda \mid \neg \delta \mid \delta \wedge \delta$). We call the logical constants $\sim, \vee, \exists^1, \forall^1$, which were introduced in [33, 1.26], *weak classical negation*, *weak disjunction*, *weak existential quantifier* and *weak universal quantifier*, respectively. Their team semantics are defined as:

- $M \models_X \sim \phi$ iff $X = \emptyset$ or $M \not\models_X \phi$.
- $M \models_X \phi \vee \psi$ iff $M \models_X \phi$ or $M \models_X \psi$.
- $M \models_X \exists^1 x \phi$ iff $M \models_{X(a/x)} \phi$ for some $a \in M$, where $X(a/x) = \{s(a/x) \mid s \in X\}$.
- $M \models_X \forall^1 x \phi$ iff $M \models_{X(a/x)} \phi$ for all $a \in M$.

It is easy to verify that formulas of **FOT** and **FOT[↓]** have the empty team property, and **FOT[↓]** formulas have the downward closure property.

In **FOT** we adopt the usual convention for classical implication, and write $\phi \rightarrow \psi$ for $\sim \phi \vee \psi$ and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. With the help of the classical implication $\rightarrow$, one can easily express $\neg \delta$ for δ being quantifier-free and disjunction free (or flat) as

$$\neg \delta(x) \equiv \forall^1 y (y \subseteq x \rightarrow \sim \delta(y/x)).$$

Also, dependence atoms $=(x, y)$ with variables as arguments are definable in **FOT**:

$$=(x, y) \equiv \forall^1 u_0 v_0 u_1 v_1 ((u_0 v_0 \subseteq xy \wedge u_1 v_1 \subseteq xy \wedge u_0 = u_1) \rightarrow v_0 = v_1).$$

Recall that the *constancy atom* $=(x)$ abbreviates the dependence atom $=(\langle \rangle, x)$ with the first argument being the empty sequence $\langle \rangle$, and its semantics reduces to

- $M \models_X =(x)$ iff for all $s, s' \in X$, $s(x) = s'(x)$.

Constancy atoms can be defined alternatively as $=(x) \equiv \exists^1 y (y = x)$.

3 Characterizing first-order team properties

In this section, we prove that **FOT**-formulas characterize first-order team properties (modulo the empty team), and **FOT**[↓]-formulas characterize downward closed first-order team properties (modulo the empty team).

Let us first define formally the relevant notions. Observe that a team X of an $\mathcal{L}$ -model M over a domain $\{v_1, \dots, v_k\}$ can also be viewed as a k -ary relation $rel(X) \subseteq M^k$ defined as $rel(X) = \{(s(v_1), \dots, s(v_k)) \mid s \in X\}$. We call a collection $\mathcal{P}_M \subseteq \wp(M^k)$ of k -ary relations (or teams) of an $\mathcal{L}$ -model M a *local team property*; and a (*global*) *team property* is a class $\mathcal{P}$ of local team properties $\mathcal{P}_M$ for all $\mathcal{L}$ -models M . A formula $\phi(\mathbf{v})$ of a logic based on team semantics clearly defines a team property $\mathcal{P}^\phi$ such that for all M , $\mathcal{P}_M^\phi = \{rel(X) \mid M \models_X \phi(\mathbf{v})\}$. Clearly, the team properties $\mathcal{P}^\phi$ defined by **FOT**[↓]-formulas ϕ are *downward closed*, that is, $A \subseteq B \in \mathcal{P}_M^\phi$ implies $A \in \mathcal{P}_M^\phi$ for any M . Note also that while the empty relation $\emptyset$ may not be contained in a team property $\mathcal{P}_M$, since all team-based logics considered in this paper have the empty team property (i.e., $\emptyset$ is in $\mathcal{P}_M^\phi$ for all ϕ), we will confine ourselves only to those team properties $\mathcal{P}$ with the empty relation $\emptyset$ contained in each local property $\mathcal{P}_M$.

We call a team property $\mathcal{P}$ *first-order* if there is a first-order $\mathcal{L}(R)$ -sentence $\alpha(R)$ such that $(M, A) \models \alpha(R)$ iff $A \in \mathcal{P}_M$ for all M and all nonempty relations A . It is worth noting that we are using the terminology “definability” in two different semantic settings: Even though every first-order team property $\mathcal{P}$ is (trivially) defined by some first-order $\mathcal{L}(R)$ -sentence $\alpha(R)$ with an extra relation symbol R (in the sense of the usual semantics of first-order logic), it does not follow that each such first-order team property $\mathcal{P}$ is definable by some first-order $\mathcal{L}$ -formula $\beta(\mathbf{v})$ in the team semantics sense (i.e., $\mathcal{P} = \mathcal{P}_M^\beta$). As a simple illustration, in view of the flatness property of first-order formulas, the following very simple team property (of the empty vocabulary $\mathcal{L}_0$)

$$\mathcal{P}_{\leq 1} = \{(M, rel(X)) \mid M \text{ an } \mathcal{L}_0\text{-model and } |X| \leq 1\}$$

cannot be defined by any first-order formula $\beta(\mathbf{v})$ of the empty vocabulary $\mathcal{L}_0$.

We now show that the team properties defined by formulas of **FOT** and **FOT**[↓] are first-order.

Theorem 1. *For any $\mathcal{L}$ -formula $\phi(v_1, \dots, v_k)$ of **FOT** or **FOT**[↓], there exists a first-order $\mathcal{L}(R)$ -sentence $\gamma_\phi(R)$ with a fresh k -ary relation symbol R such that for any $\mathcal{L}$ -model M and any team X over $\{v_1, \dots, v_k\}$,*

$$M \models_X \phi \iff (M, rel(X)) \models \gamma_\phi(R). \quad (2)$$

Proof. We prove the theorem by proving a slightly more general claim: For any subformula $\theta(\mathbf{v}, \mathbf{x})$ of $\phi(\mathbf{v})$, there exists a first-order $\mathcal{L}(R)$ -formula $\gamma_\theta(R, \mathbf{x})$ such that for all $\mathcal{L}$ -models M , teams X and sequences $\mathbf{a}$ of elements in M ,

$$M \models_{X(\mathbf{a}/\mathbf{x})} \theta(\mathbf{v}, \mathbf{x}) \iff (M, rel(X)) \models \gamma_\theta(R, \mathbf{x})(\mathbf{a}/\mathbf{x}).$$

It is easy to verify that the formula γ_θ defined inductively as follows (and found essentially in, e.g., [11,33]) will work:

- If $\theta(v, x) = \delta(v, x)$ for some quantifier-free and disjunction-free first-order formula δ , let $\gamma_\theta(R, x) = \forall u(Ru \rightarrow \delta(u/v, x))$.
- If $\theta(v, x) = \rho(vx) \subseteq \sigma(vx)$, where $\rho(vx)$ and $\sigma(vx)$ are two sequences of variables from vx , let $\gamma_\theta(R, x) = \forall u \exists w(Ru \rightarrow (Rw \wedge \rho(ux) = \sigma(wx)))$.
- If $\theta(v, x) = \theta_0(v, x) \wedge \theta_1(v, x)$ and does not belong to the case of the first item, let $\gamma_\theta(R, x) = \gamma_{\theta_0}(R, x) \wedge \gamma_{\theta_1}(R, x)$.
- If $\theta(v, x) = \theta_0(v, x) \vee \theta_1(v, x)$, let $\gamma_\theta(R, x) = \gamma_{\theta_0}(R, x) \vee \gamma_{\theta_1}(R, x)$.
- If $\theta(v, x) = \sim \theta_0(v, x)$, let $\gamma_\theta(R, x) = \forall u \neg Ru \vee \neg \gamma_{\theta_0}(R, x)$.
- If $\theta(v, x) = \exists^1 y \theta_0(v, yx)$, let $\gamma_\theta(R, x) = \exists y \gamma_{\theta_0}(R, yx)$.
- If $\theta(v, x) = \forall^1 y \theta_0(v, yx)$, let $\gamma_\theta(R, x) = \forall y \gamma_{\theta_0}(R, yx)$. $\square$

Next we prove the reverse direction of Theorem 1, from which we can conclude that **FOT**-formulas characterize exactly first-order team properties (modulo the empty team).

Theorem 2. *For any first-order $\mathcal{L}(R)$ -sentence $\gamma(R)$ with a k -ary relation symbol R , there exists an $\mathcal{L}$ -formula $\phi_\gamma(v_1, \dots, v_k)$ of **FOT** such that for any $\mathcal{L}$ -model M and any nonempty team X over $\{v_1, \dots, v_k\}$,*

$$M \models_X \phi_\gamma(v) \iff (M, \text{rel}(X)) \models \gamma(R).$$

Moreover, if R occurs in $\gamma(R)$ only negatively (i.e., every occurrence of R is in the scope of an odd number of nested negation symbols), ϕ_γ can be also chosen to be an **FOT**[↓]-formula.

Proof. We may assume w.l.o.g. that the first-order sentence $\gamma(R)$ is in prenex normal form $Q_1 x_1 \dots Q_n x_n \theta(x)$, where $Q_i \in \{\forall, \exists\}$, θ is quantifier-free and in negation normal form (i.e., negations occur only in front of atomic formulas), and every occurrence of R is of the form Rx_i for some sequence x_i of bound variables (for $Rt \equiv \exists y(y = t \wedge Ry)$).

Define the translation $\phi_\gamma(v) := Q_1^1 x_1 \dots Q_n^1 x_n \phi_\theta(x, v)$ in **FOT**, where $Q_i^1 = \forall^1$ if $Q_i = \forall$, $Q_i^1 = \exists^1$ if $Q_i = \exists$, and $\phi_\theta(x, v)$ is defined inductively as follows:

- if $\theta = \lambda(x)$ is an atomic formula in which R does not occur, then $\phi_\lambda(x, v) = \lambda(x)$;
- if $\theta = Rx_i$, then $\phi_\theta(x, v) = x_i \subseteq v$;
- if $\theta = \neg \lambda$ for some (atomic) formula λ , then $\phi_\theta = \sim \phi_\lambda$;
- if $\theta = \theta_0 \wedge \theta_1$, then $\phi_\theta = \phi_{\theta_0} \wedge \phi_{\theta_1}$;
- if $\theta = \theta_0 \vee \theta_1$, then $\phi_\theta = \phi_{\theta_0} \vee \phi_{\theta_1}$.

Next, we show by induction that for each quantifier-free formula $\theta(x)$, for any nonempty team X over $\{v_1, \dots, v_k\}$ and $a_1, \dots, a_n \in M$,

$$M \models_{X(a/x)} \phi_\theta(x, v) \iff (M, \text{rel}(X)) \models \theta(a/x). \quad (3)$$

If $\theta = \lambda(x)$ is an atomic formula in which R does not occur, then $\phi_\lambda = \lambda(x)$ and

$$\begin{aligned} M \models_{X(a/x)} \lambda(x) &\iff M \models_X \lambda(x)(a/x) \\ &\iff M \models_s \lambda(x)(a/x) \text{ for all } s \in X \\ &\iff (M, \text{rel}(X)) \models \lambda(a/x). \quad (\text{since } R \text{ does not occur in } \lambda) \end{aligned}$$

If $\theta = Rx_i$, then $\phi_\theta = x_i \subseteq v$ and

$$\begin{aligned} M \models_{X(a/x)} x_i \subseteq v &\iff \text{For all } s \in X(a/x), \text{ there exists } s' \in X(a/x) \text{ s.t. } s(x_i) = s'(v) \\ &\iff a_i \in \text{rel}(X) = \{s'(v) \mid s' \in X\} \\ &\iff (M, \text{rel}(X)) \models Rx_i(a/x). \end{aligned}$$

If $\theta = \neg\lambda(x)$, then

$$\begin{aligned} M \models_{X(a/x)} \sim\phi_\lambda(x, v) &\iff M \not\models_{X(a/x)} \phi_\lambda(x, v) && (\text{since } X(a/x) \neq \emptyset) \\ &\iff (M, \text{rel}(X)) \not\models \lambda(a/x) && (\text{by induction hypothesis}) \\ &\iff (M, \text{rel}(X)) \models \neg\lambda(a/x). \end{aligned}$$

The cases when $\theta = \theta_0(x) \vee \theta_1(x)$ and $\theta = \theta_0(x) \wedge \theta_1(x)$ follow easily from the induction hypothesis.

Finally, we have

$$\begin{aligned} M \models_X Q_1^1 x_1 \dots Q_n^1 x_n \phi_\theta(x, v) &\iff Q_1 a_1 \in M \dots Q_n a_n \in M : M \models_{X(a/x)} \phi_\theta(x, v) \\ &\iff Q_1 a_1 \in M \dots Q_n a_n \in M : (M, \text{rel}(X)) \models \theta(a/x) \\ &\iff (M, \text{rel}(X)) \models Q_1 x_1 \dots Q_n x_n \theta(x). \end{aligned}$$

This completes the proof for the translation into **FOT**. Now, if R occurs only negatively in γ (thus also in θ), we can define alternatively the translation into **FOT**[↓] as: If $\theta = \neg Rx_i$, define $\phi_\theta = \neg \bigwedge_j x_{ij} = v_j$; if $\theta = \alpha(x)$ is a literal in which R does not occur, define $\phi_\alpha = \alpha(x)$. It is easy to verify that (3) still holds for these two cases. $\square$

To conclude from the above theorems that **FOT**[↓]-formulas characterize downward closed first-order team properties (modulo the empty team), we now prove a characterization theorem for first-order sentences $\alpha(R)$ that define downward closed team properties, by applying Lyndon's Interpolation Theorem of first-order logic, which we recall below.

Theorem 3 (Lyndon's Interpolation [29]). *Let α be a first-order $\mathcal{L}_0$ -formula and β a first-order $\mathcal{L}_1$ -formula. If $\alpha \models \beta$, then there is a first-order $\mathcal{L}_0 \cap \mathcal{L}_1$ -formula δ such that $\alpha \models \delta$ and $\delta \models \beta$, and moreover a predicate symbol has a positive (resp. negative) occurrence in δ only if it has a positive (resp. negative) occurrence in both α and β .*

Proposition 4. *A first-order $\mathcal{L}(R)$ -sentence $\alpha(R)$ defines a downward closed team property with respect to R if and only if there is a first-order $\mathcal{L}(R)$ -sentence $\beta(R)$ such that $\alpha \equiv \beta$ and R occurs only negatively in β .*

Proof. “ $\Leftarrow$ ”: Suppose α is a first-order $\mathcal{L}(R)$ -sentence in which the k -ary predicate R occurs only negatively, and we assume w.l.o.g. that α is in negation normal form. We can show by induction that α is downwards closed with respect to R . The only nontrivial case is when $\alpha = \neg Rt$. In this case, for any model M , any $A \subseteq B \subseteq M^k$, and any assignment s , $(M, B) \models_s \neg Rt \implies t^M \langle s \rangle \notin B \implies t^M \langle s \rangle \notin A \implies (M, A) \models_s \neg Rt$.

“ $\Rightarrow$ ”: Suppose that α is a first-order $\mathcal{L}(R)$ -sentence that is downwards closed with respect to R . It is easy to see that $\alpha \equiv \exists S(\alpha(S/R) \wedge \forall x(Rx \rightarrow Sx))$, where $\alpha(S/R)$ is

Table 1. Introduction and elimination rules for the weak logical constants

$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge I$	$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \wedge E \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} \wedge E$
$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \vee I \quad \frac{\Gamma \vdash \phi}{\Gamma \vdash \psi \vee \phi} \vee I$	$\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \vee E$
$\frac{\Gamma \vdash \phi(t/x) \quad \Gamma \vdash =(t)}{\Gamma \vdash \exists^1 x \phi} \exists^1 I$	$\frac{\Gamma \vdash \exists^1 x \phi \quad \Gamma, \phi(v/x), =(v) \vdash \psi}{\Gamma \vdash \psi} \exists^1 E \text{ (a)}$
	$\frac{\Gamma \vdash \exists^1 x \phi \quad \Gamma, \phi(c/x) \vdash \psi}{\Gamma \vdash \psi} \exists^1 E \text{ (b)}$
$\frac{\Gamma \vdash \phi(v/x) \quad \Gamma \vdash =(v)}{\Gamma \vdash \forall^1 x \phi} \forall^1 I \text{ (c)}$	$\frac{\Gamma \vdash \forall^1 x \phi \quad \Gamma \vdash =(t)}{\Gamma \vdash \phi(t/x)} \forall^1 E$
$\frac{\Gamma \vdash \phi(c/x)}{\Gamma \vdash \forall^1 x \phi} \forall^1 I \text{ (d)}$	
(a). $v \notin \text{Fv}(\Gamma \cup \{\phi, \psi\})$ (b). c does not occur in Γ, ϕ, ψ (c). $v \notin \text{Fv}(\Gamma \cup \{\phi\})$ (d). c does not occur in Γ, ϕ	

obtained from α by replacing every occurrence of R by S . Put $\gamma = \alpha(S/R) \wedge \forall x(Rx \rightarrow Sx)$, and note that R occurs only negatively in γ . Then, $\gamma \models \alpha$, since for any $\mathcal{L}(R, S)$ -model (M, A, B) such that $(M, A, B) \models \gamma(R, S)$, we have $(M, A) \models \exists S\gamma(R, S)$, which implies $(M, A) \models \alpha(R)$.

Now, by Lyndon's Interpolation Theorem, there is a first-order $\mathcal{L}(R)$ -sentence β such that $\gamma(R, S) \models \beta(R)$ and $\beta(R) \models \alpha(R)$, and moreover, R occurs only negatively in β . It remains to show $\alpha \models \beta$. For any $\mathcal{L}(R)$ -model (M, A) such that $(M, A) \models \alpha(R)$. Clearly the $\mathcal{L}(R, S)$ -model (M, A, A) satisfies $(M, A, A) \models \alpha(S/R) \wedge \forall x(Rx \rightarrow Sx)$. Since $\gamma \models \beta$, we have $(M, A, A) \models \beta(R)$, thereby $(M, A) \models \beta(R)$. $\square$

Corollary 5. For any $\mathcal{L}$ -formula $\phi(v)$ of $\mathbf{FOT}^\downarrow$, there exists a first-order $\mathcal{L}(R)$ -sentence $\gamma_\phi(R)$ with R occurring only negatively such that (2) holds, and vice versa. In particular, $\mathbf{FOT}^\downarrow$ -formulas characterize exactly downward closed first-order team properties (modulo the empty team).

4 Axiomatizing FOT

In this section, we introduce a system of natural deduction for **FOT**, and prove the soundness and completeness theorem. For the convenience of our proofs, we present our system of natural deduction in sequent style.

Definition 6. The system of natural deduction for **FOT** consists of all rules for identity, all rules in Table 1, and the following rules, where letters in sans-serif face (such as x, y) stand for sequences of variables, c is a constant symbol, $=(t)$ is short for $\exists^1 x(x = t)$, $\text{con}(t)$ is short for $\bigwedge_i =(t_i)$, and $c\mathbf{x} \subseteq v\mathbf{y}$ is short for $\exists^1 u(u = c \wedge u\mathbf{x} \subseteq v\mathbf{y})$:

$\frac{\phi \in \Gamma}{\Gamma \vdash \phi} \text{Assml}$	
$\frac{\Gamma, \phi \vdash \perp}{\Gamma \vdash \sim \phi} \sim I$	$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \sim \phi}{\Gamma \vdash \psi} \sim E \quad \frac{\Gamma, \sim \phi \vdash \perp}{\Gamma \vdash \phi} \text{RAA}$
$\frac{}{\Gamma \vdash \text{=(c)}} \text{conI}$	$\frac{\Gamma \vdash \text{con(t)}}{\Gamma \vdash \text{=(ft)}} \text{conI}$
$\frac{}{\Gamma \vdash x \subseteq x} \subseteq \text{Id}$	$\frac{\Gamma \vdash x_1 \dots x_n \subseteq y_1 \dots y_n}{\Gamma \vdash x_{i_1} \dots x_{i_k} \subseteq y_{i_1} \dots y_{i_k}} \subseteq \text{Pro (a)}$
$\frac{\Gamma \vdash x \subseteq y \quad \Gamma \vdash y \subseteq z}{\Gamma \vdash x \subseteq z} \subseteq \text{Tr}$	$\frac{\Gamma \vdash x \subseteq y \quad \Gamma \vdash \alpha(y)}{\Gamma \vdash \alpha(x)} \subseteq \text{Cmp (b)}$
$\frac{\Gamma \vdash \text{con}(x) \quad \Gamma \vdash y \subseteq z}{\Gamma \vdash xy \subseteq xz} \subseteq W_{\text{con}}$	$\frac{\Gamma \vdash \text{con}(x) \quad \Gamma \vdash x \subseteq y}{\Gamma \vdash \exists^1 z (zx \subseteq wy)} \subseteq W_{\exists^1}$
$\frac{\Gamma \vdash \sim x \subseteq y \quad \Gamma, c \subseteq x, \sim c \subseteq y \vdash \phi}{\Gamma \vdash \phi} \sim \subseteq E \text{ (c)}$	
$\frac{\Gamma \vdash \sim \lambda(x) \quad \Gamma, c \subseteq x, \sim \lambda(c) \vdash \phi}{\Gamma \vdash \phi} \sim \lambda E \text{ (c)}$	
$\frac{\Gamma, \exists^1 z Rz, \phi(R) \vdash \perp}{\Gamma, \phi(v) \vdash \perp} \subseteq \text{wl}_R \text{ (d)}$	
<p>(a). $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ (b). α is $\sim$ and inclusion atom-free. (c). c is a sequence of constant symbols that do not occur in Γ or ϕ, and λ is a first-order atom. (d). Γ is a set of sentences in which R does not occur, $\phi(R)$ is an inclusion atom-free sentence in which the relation symbol R occurs only in the form Rx, and $\phi(v)$ is a formula with free variables v obtained from $\phi(R)$ by replacing every Rx by $x \subseteq v$.</p>	

We write $\Gamma \vdash_{\text{FOT}} \phi$ or simply $\Gamma \vdash \phi$, if the sequent $\Gamma \vdash \phi$ is derivable in the system. Write $\phi \dashv\vdash \psi$ if $\phi \vdash \psi$ and $\psi \vdash \phi$.

The weak disjunction $\vee$ admits the usual introduction and elimination rule, and note that the usual elimination rule is not sound for the other disjunction $\vee$. The soundness of the introduction and elimination rule for $\exists^1$ follows from the equivalence $\exists^1 x \equiv \exists x (\text{=(}x) \wedge \phi)$, and the introduction and elimination rule for $\forall^1$ have a similar flavor. The rules $\subseteq \text{Id}$, $\subseteq \text{Pro}$, $\subseteq \text{Tr}$ and $\subseteq \text{Cmp}$ for inclusion atoms were introduced in [17], and the first three rules completely axiomatize the implication problem of inclusion dependencies in database theory [3]. The two weakening rules $\subseteq W_{\text{con}}$ and $\subseteq W_{\exists^1}$ for inclusion atoms extend the length of an inclusion atom. We leave it for the reader to verify that these rules for inclusion atoms are also sound and derivable if constants are allowed to occur as arguments in inclusion atoms (i.e., to allow inclusion atoms, e.g., of the form $cx \subseteq vy$). The rules $\sim \subseteq E$ and $\sim \lambda E$ in a sense describe the meanings

of a negated inclusion atom $\sim x \subseteq y$ and a negated first-order atom $\lambda(x)$ by providing a witness c . These two rules are designed for deriving Proposition 9(ii)(v) (which is crucial for the normal form lemma, Lemma 11, leading to the completeness theorem), and they can also be formulated, in a more complex form, without any mention of the constant symbols. The rule $\subseteq \text{wl}_R$ simulates the transformation in Theorem 2, and it will be applied in the proof of the completeness theorem (Theorem 13) in a reverse manner with respect to a fresh relation symbol R , which is assumed to be always available. How to simplify this rule $\subseteq \text{wl}_R$ is left as future work.

Theorem 7 (Soundness). $\Gamma \vdash_{\text{FOT}} \phi \implies \Gamma \models \phi$.

Proof. We only verify the soundness of $\sim \subseteq \text{E}$ and $\subseteq \text{wl}_R$.

$\sim \subseteq \text{E}$: Suppose $\Gamma \models \sim x \subseteq y$ and $\Gamma, c \subseteq x, \sim c \subseteq y \models \phi$, and suppose that for some $\mathcal{L}$ -model M and team X , $M \models_X \Gamma$. Then we have $M \models_X \sim x \subseteq y$, which implies that there exists $s \in X$ such that for the $\mathcal{L}(c)$ -model $(M, s(x))$, we have $(M, s(x)) \models_X c \subseteq x \wedge \sim c \subseteq y$. Thus, by the assumption, $(M, s(x)) \models_X \phi$, which gives $M \models_X \phi$ since c do not occur in ϕ .

$\subseteq \text{wl}_R$: Suppose $\Gamma, \phi(v) \not\models \perp$. Clearly, every **FOT**-formula can be turned into a (semantically) equivalent formula in prenex and negation normal form (cf. Proposition 8(ii)(iii)(iv)). We may then w.l.o.g. assume that $\phi(v)$ is in prenex and negation normal form. Then there exist a model M and a nonempty team X such that $M \models_X \Gamma$ and $M \models_X \phi(v)$. By (the proof of) Theorem 2, $(M, \text{rel}(X)) \models \phi_*(R)$ in **FO**, where $\phi_*(R)$ is an **FO**-sentence obtained from the inclusion atom-free **FOT**-sentence $\phi(R)$ by replacing every logical constant in **FOT** by its counterpart in **FO**, i.e., by replacing $\sim$ by $\neg$, $\forall$ by $\exists$, $\exists$ by $\forall$, and $\exists^1$ by $\exists$. It is not hard to prove that $(M, \text{rel}(X)) \models_{\{\emptyset\}} \phi(R)$ in **FOT** follows. Since Γ is a set of sentences in which R does not occur, we also have $(M, \text{rel}(X)) \models_{\{\emptyset\}} \Gamma$. Also, since $X \neq \emptyset$, $(M, \text{rel}(X)) \models \exists^1 z R z$. Hence, we conclude $\Gamma, \exists^1 z R z, \phi(R) \not\models \perp$. $\square$

We collect the basic facts concerning the logical constants in **FOT** in the following proposition. The proofs are standard and left to the reader.

- Proposition 8.** (i) $\Gamma, \forall^1 x \phi \vdash \phi(c/x)$ and $\Gamma, \phi(c/x) \vdash \exists^1 x \phi$.
(ii) $Q^1 x \phi \wedge \psi \dashv\vdash Q^1 x (\phi \wedge \psi)$ and $Q^1 x \phi \vee \psi \dashv\vdash Q^1 x (\phi \vee \psi)$, whenever $x \notin Fv(\psi)$.
(iii) $\Gamma, \phi \vdash \psi$ iff $\Gamma, \sim \psi \vdash \sim \phi$, and $\Gamma, \sim \sim \phi \vdash \phi$.
(iv) $\sim \forall^1 x \phi \dashv\vdash \exists^1 x \sim \phi$, $\sim \exists^1 x \phi \dashv\vdash \forall^1 x \sim \phi$,
 $\sim(\phi \vee \psi) \dashv\vdash \sim \phi \wedge \sim \psi$ and $\sim(\phi \wedge \psi) \dashv\vdash \sim \phi \vee \sim \psi$.

A routine inductive proof that uses Proposition 8(i) shows that the usual Replacement Lemma holds for our logic, that is, if $\theta \dashv\vdash \chi$, then $\phi \dashv\vdash \phi(\chi/\theta)$, where $\phi(\chi/\theta)$ is obtained from ϕ by replacing an occurrence of θ in ϕ by χ .

It is easy to prove that $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \rightarrow \psi$. In the following proposition, we list some derivable technical clauses that will be used in the proof of the completeness theorem. See Appendix for the proof.

Proposition 9. Let ξ and η be two sequences of variables of the same length.

- (i) $xy\xi \subseteq v\eta \dashv\vdash x = y \wedge x\xi \subseteq v\eta$.

- (ii) $\xi \subseteq \eta \Vdash \forall^1 x (x \subseteq \xi \rightarrow x \subseteq \eta)$.
- (iii) $\text{con}(z) \vdash w\xi \subseteq z\eta \leftrightarrow (w = z \wedge \xi \subseteq \eta)$.
- (iv) $\text{con}(x) \vdash x \subseteq v \leftrightarrow \exists^1 y (xy \subseteq vu)$.
- (v) If $\lambda(z)$ is a first-order atom, then $\lambda(z) \Vdash \forall^1 w (w \subseteq z \rightarrow \lambda(w))$.

To prove the completeness theorem, we also need the following three lemmas. The first lemma emphasizes the fact that all variables quantified by the weak quantifiers have constant values, the second lemma proves a normal form for **FOT**-formulas, and the third lemma shows that derivations in the system of **FO** can be simulated in the system of **FOT**.

Lemma 10. *Let $\phi(v) = Q^1 x \theta(x, v)$ be a formula in prenex and negation normal form. Then $\phi \Vdash \phi_{\text{con}}$, where ϕ_{con} is the formula obtained from ϕ by replacing every (first-order or inclusion) literal $\mu(x, v)$ (i.e., an atom or negated atom) by $\mu \wedge \text{con}(x)$.*

Proof. By applying Proposition 8(ii), $Q^1 I$ and $Q^1 E$, it is easy to prove that $Q^1 x \theta(x, v) \Vdash Q^1 x (\theta(x, v) \wedge \text{con}(x))$. Next we push the formula $\text{con}(x)$ inside the quantifier-free formula θ in negation normal form all the way to the front of literals by using Replacement Lemma and the standard equivalences $(\theta_0 \wedge \theta_1) \wedge \text{con}(x) \Vdash (\theta_0 \wedge \text{con}(x)) \wedge (\theta_1 \wedge \text{con}(x))$ and $(\theta_0 \vee \theta_1) \wedge \text{con}(x) \Vdash (\theta_0 \wedge \text{con}(x)) \vee (\theta_1 \wedge \text{con}(x))$. $\square$

Lemma 11. *For every **FOT**-formula ϕ , we have $\phi(v) \Vdash Q^1 x \theta(x, v)$, where $\theta(x, v)$ is a quantifier-free formula in negation normal form in which first-order atoms are of the form $\lambda(x)$, and inclusion atoms are of the form $x_i \subseteq v$ for some variables x_i from x .*

Proof. We first turn $\phi(v)$ into an equivalent formula in prenex and negation normal form by exhaustively applying Proposition 8(ii)(iii)(iv). Assume that the bound variables of $\phi(v)$ are among x . By Lemma 10 we may also assume that every literal $\mu(x, v)$ in ϕ is replaced by $\mu(x, v) \wedge \text{con}(x)$ (call such a formula a formula in *constant normal form*). Observe that now in $\phi(v)$ a generic first-order atom is of the form $\lambda(x, v)$, and a generic inclusion atom is of the form $\eta\xi\rho\sigma \subseteq \eta'\xi'\rho'\sigma'$ (modulo permutation by $\subseteq$ Pro), where $|\eta| = |\eta'| \geq 0$, $|\xi| = |\xi'| \geq 0$, $|\rho| = |\rho'| \geq 0$ and $|\sigma| = |\sigma'| \geq 0$,

- $(\eta, \eta') = (x_i, x_j)$ for some bound variables x_i, x_j from x ;
- $(\xi, \xi') = (x_i, v_i)$ for some bound variables x_i from x , and free variables v_i from v ;
- $(\rho, \rho') = (v_i, x_i)$ for some free variables v_i from v , and bound variables x_i from x ;
- $(\sigma, \sigma') = (v_i, v_j)$ for some free variables v_i, v_j from v .

To obtain the required normal form we have to transform every (first-order or inclusion) atom in ϕ in the required form. We achieve this in several steps.

In Step 1 of our transformation, we replace in $\phi(v)$ every inclusion atom $\eta\xi v_i \sigma \subseteq \eta'\xi'x_i\sigma'$ by $v_i = x_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma'$. Note that by Proposition 9(iii), we have

$$\text{con}(x_i) \wedge \eta\xi v_i \sigma \subseteq \eta'\xi'x_i\sigma' \Vdash \text{con}(x_i) \wedge (v_i = x_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma')$$

$$\text{and } \text{con}(x_i) \wedge \sim \eta\xi v_i \sigma \subseteq \eta'\xi'x_i\sigma' \Vdash \text{con}(x_i) \wedge \sim (v_i = x_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma').$$

Hence, by Replacement Lemma, the resulting formula $\phi_1(v)$ is provably equivalent to ϕ . We assume further (here and also in the other steps) that $\phi_1(v)$ is turned into prenex, negation and constant normal form by applying Proposition 8(ii)(iii)(iv) and Lemma 10.

In Step 2, we replace in $\phi_1(v)$ every first-order atom $\lambda(x, v)$ by $\forall^1 yz(yz \subseteq xv \rightarrow \lambda(y, z))$. By Proposition 9(v), the resulting formula $\phi_2(v)$ is provably equivalent to $\phi_1(v)$. Up to now, every first-order atom in the formula is transformed to the required form, and the steps afterwards will not generate first-order atoms in non-normal form.

In Step 3, we apply Proposition 9(ii) to replace in $\phi_2(v)$ every inclusion atom $\eta\xi v_i \subseteq \eta'\xi'v_j$ by $\forall^1 wyz(wyz \subseteq \eta\xi v_i \rightarrow wyz \subseteq \eta'\xi'v_j)$, and denote the resulting formula by $\phi_3(v)$. In Step 4, we apply Proposition 9(iii) to replace in $\phi_3(v)$ every inclusion atom $x_i x_k \subseteq x_j v_k$ by $x_i = x_j \wedge x_k \subseteq v_k$, and denote the resulting formula by $\phi_4(v)$. Up to now every inclusion atom in the formula is transformed to the form $x_i \subseteq v_i$, where x_i are bound variables and v_i are free variables in $\phi_4(v)$. Yet, v_i may contain repetitions, and it may also be only a subsequence of v . Handling these requires two additional steps.

In Step 5, we remove repetitions on the right side of the inclusion atoms, by applying Proposition 9(i) to replace in $\phi_4(v)$ every inclusion atom of the form $x_i x_j x_k \subseteq v_i v_j v_j$ by $x_i = x_j \wedge x_i x_k \subseteq v_i v_j$. Denote the resulting formula by $\phi_5(v)$. In Step 6, we extend the length of those shorter inclusion atoms. Assuming $v = v_i v_j$, we apply Proposition 9(iv) to replace in $\phi_5(v)$ every inclusion atom of the form $x_i \subseteq v_i$ by $\exists^1 y(x_i y \subseteq v_i v_j)$. Denote the resulting formula by $\phi_6(v)$.

As before we assume that $\phi_6(v)$ is turned into prenex and negation normal form, but now we shall apply Lemma 10 in a reverse manner to remove the added constancy atoms for each literal in ϕ_6 . Finally, the resulting provably equivalent formula is in the required normal form. Note that our transformation clearly terminates, because we have performed the steps in the transformation in such an order that each step will not generate new formulas for which the transformations in the previous steps apply. $\square$

Lemma 12. *Let $\Delta \cup \{\delta\}$ be a set of **FO**-formulas whose free variables are among x . If $\Delta \vdash_{\mathbf{FO}} \delta$, then $\Delta^*(c/x) \vdash_{\mathbf{FOT}} \delta^*(c/x)$, where $*$ is the operation that replaces every logical constant in **FO** by its counterpart in **FOT**, and c is a sequence of fresh constant symbols. In particular, if $\Delta \cup \{\delta\}$ is a set of **FO**-sentences, then $\Delta \vdash_{\mathbf{FO}} \delta$ implies $\Delta^* \vdash_{\mathbf{FOT}} \delta^*$.*

Proof. See Appendix.

Finally, we are in a position to prove the completeness theorem of our system.

Theorem 13 (Completeness). $\Gamma \models \phi \implies \Gamma \vdash_{\mathbf{FOT}} \phi$.

Proof. Since **FOT** (being expressively equivalent to **FO**) is compact, we may assume that Γ is finite. Now, suppose $\Gamma \not\vdash_{\mathbf{FOT}} \phi$. By RAA we derive $\Gamma, \sim \phi \vdash_{\mathbf{FOT}} \perp$, which is equivalent to $Q^1 x \theta(x, v) \vdash_{\mathbf{FOT}} \perp$, where $\psi(v) = Q^1 x \theta(x, v)$ is the normal form of the formula $\bigwedge \Gamma \wedge \sim \phi$ given by Lemma 11. By applying $\subseteq w l_R$ we obtain $\exists^1 z R z, \psi(R) \vdash_{\mathbf{FOT}} \perp$, where R is a fresh relation symbol, and $\psi(R)$ is the inclusion atom-free sentence obtained from $\psi(v)$ by replacing every inclusion atom $x_i \subseteq v$ by $R x_i$. It follows from Lemma 12 that $\exists z R z, \psi_*(R) \vdash_{\mathbf{FO}} \perp$, where ψ_* is the **FO**-formula obtained from the **FOT**-formula ψ by replacing every logical constant in **FOT** by its counterpart in **FO**. Now, by the completeness theorem of **FO**, there exists a model (M, R^M) such that $R^M \neq \emptyset$ and $(M, R^M) \models \psi_*(R)$. It then follows from (the proof of) Theorem 2 that $M \models_{X_R} \psi(v)$, where $X_R = \{s : \{v_1, \dots, v_k\} \rightarrow M \mid s(v) \in R^M\}$ is the (nonempty) team associated with R . Hence $\psi \not\vdash \perp$. $\square$

5 Axiomatizing $\mathbf{FOT}^\downarrow$ consequences in $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$

Let $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ denote dependence logic ($\mathbf{D}$) extended with the syntax of $\mathbf{FOT}^\downarrow$, that is, formulas of $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ are defined by the grammar:

$$\phi ::= \lambda \mid =(t_1, \dots, t_n, t) \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \wp \phi \mid \exists x\phi \mid \forall x\phi \mid \exists^1 x\phi \mid \forall^1 x\phi$$

where λ is a first-order atom and α is first-order. Recall that $\mathbf{D}$ captures downward closed **ESO** team properties [26]. It can be easily seen from the proof of Theorem 1 that enriching the syntax of $\mathbf{D}$ with the weak connective $\wp$ and quantifiers $\exists^1, \forall^1$ from $\mathbf{FOT}^\downarrow$ does not increase the expressive power of the logic; in other words, $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ has the same expressive power as $\mathbf{D}$. In this section, we introduce a system of natural deduction for the logic $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ by extending the systems of [27] and [33] for $\mathbf{D}$ so that this new system is sound and complete for $\mathbf{FOT}^\downarrow$ consequences in the sense that

$$\Gamma \models \theta \iff \Gamma \vdash \theta$$

whenever Γ is a set of sentences in $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$, and θ is a sentence in $\mathbf{FOT}^\downarrow$.

One crucial step in our argument for the completeness theorem involves an application of the rule

$$\frac{\Gamma, \sim\phi \vdash \perp}{\Gamma \vdash \phi} \text{RAA},$$

where, as in [33], the formula $\sim\phi$ should be read as a shorthand for the defining formula of $\sim\phi$ in the language of the logic in question. In our system this rule will only be applied for ϕ being an $\mathbf{FOT}^\downarrow$ -sentence, and in this case $\sim\phi \equiv \neg\phi$, where $\neg\phi$ is the syntactic negation obtained by pushing negation to the very front of first-order atoms in ϕ using the definitions: $\neg\neg\lambda := \lambda$, $\neg(\psi \wedge \chi) := \neg\psi \wp \neg\chi$, $\neg(\psi \wp \chi) := \neg\psi \wedge \neg\chi$, $\neg\forall^1 x\psi := \exists^1 x\neg\psi$, $\neg\exists^1 x\psi := \forall^1 x\neg\psi$.

Definition 14. The system of natural deduction for $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ consists of all rules of the system of $\mathbf{D}$ defined in [27] (including rules of identity, and particularly those rules in Table 2), the rule RAA for ϕ in the rule being an $\mathbf{FOT}^\downarrow$ -sentence, all rules in Table 1 from Section 4, and the following rules, where α ranges over first-order formulas only:

$\frac{}{\Gamma \vdash \exists x \exists y (x \neq y)} \text{Dom}$		
$\frac{\Gamma \vdash \phi \vee \perp}{\Gamma \vdash \phi} \perp \vee \text{E}$	$\frac{\Gamma \vdash \text{con}(x)}{\Gamma \vdash \alpha(x) \wp \neg\alpha(x)} \wp \text{wl}$	$\frac{\Gamma \vdash \forall^1 x \alpha}{\Gamma \vdash \forall x \alpha} \forall^1 \forall \text{Trs}$
$\frac{\Gamma \vdash \forall^1 x \phi \vee \psi}{\Gamma \vdash \exists y \forall^1 x ((y = z \wedge \phi) \vee (y \neq z \wedge \psi))} \forall^1 \text{Ext} \quad [x \notin \text{Fv}(\psi), y, z \text{ are fresh}]$		
$\frac{\Gamma, \text{con}(x) \vdash =(y)}{\Gamma \vdash =(x, y)} =(\cdot) \text{wl}$	$\frac{\Gamma \vdash =(x, y) \quad \Gamma \vdash \text{con}(x)}{\Gamma \vdash =(y)} =(\cdot) \text{wE}$	

Table 2. Some rules from the system [27] of **D**

$\frac{\Gamma \vdash \exists x \forall y \phi(x, y, z)}{\Gamma \vdash \forall y \exists x (\phi(x, y, z) \wedge \phi)} =(\cdot)I$	$\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \chi}{\Gamma \vdash \chi \vee \psi} \vee \text{Sub}$
$\frac{\Gamma \vdash \phi(t/x)}{\Gamma \vdash \exists x \phi} \exists I$	$\frac{\Gamma \vdash \exists x \phi \quad \Gamma, \phi(v/x) \vdash \psi}{\Gamma \vdash \psi} \exists E [v \notin \text{Fv}(\Gamma \cup \{\phi, \psi\})]$

The axiom Dom stipulates that the domain of a model has at least two elements, which we assume throughout this section. This domain assumption is often postulated in the literature on dependence logic, especially because over models with singleton domain all dependence atoms become trivially true (as there is only one single assignment over such a domain). In our setting, the axiom Dom is required for Proposition 16(v), which shows that the weak disjunction $\vee$ is definable in terms of the other disjunction $\vee$ in **D** (as long as the domain has more than one elements). The rules $\perp \vee E$, $\vee wI$ and $\forall^1 \forall \text{Trs}$ are evident. The invertible rule $\forall^1 \text{Ext}$ is an adaption of a similar rule in the system of **D** in [27], and it is inspired also by a similar equivalence given in [14]. The rules $=(\cdot)wI$ and $=(\cdot)wE$ for dependence atoms were introduced in [34] in the propositional context.

Theorem 15 (Soundness). $\Gamma \vdash \phi \implies \Gamma \models \phi$.

Proof. See Appendix. $\square$

In the following proposition we list some technical clauses that will be used in our proof of the completeness theorem. See Appendix for the proof. In addition, Proposition 8(i)(ii) are still derivable in the system of **D** $\oplus$ **FOT**[↓] by the same derivation.

Proposition 16. (i) $\vdash =(\mathbf{z}, c)$, and in particular $\vdash =(\mathbf{c})$ for any constant symbol c .
(ii) $=(\mathbf{c}\mathbf{x}, \mathbf{y}) \dashv\vdash =(\mathbf{x}, \mathbf{y})$ for any constant symbol c .
(iii) $\forall^1 v Qu(\phi \wedge =(\mathbf{v}\mathbf{x}, \mathbf{y})) \dashv\vdash \forall^1 v Qu(\phi \wedge =(\mathbf{x}, \mathbf{y}))$ and $\forall^1 v Qu(\phi \wedge =(\mathbf{x}, \mathbf{v})) \dashv\vdash \forall^1 v Qu\phi$.
(iv) $\exists^1 x \phi \dashv\vdash \exists x (=(\mathbf{x}) \wedge \phi)$.
(v) $\phi \vee \psi \dashv\vdash \exists x \exists y (=(\mathbf{x}) \wedge =(\mathbf{y}) \wedge ((x = y \wedge \phi) \vee (x \neq y \wedge \psi)))$, where x, y are fresh.
(vi) $\exists x \forall^1 y \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}) \vdash \forall^1 y \exists x (=(\mathbf{z}, \mathbf{x}) \wedge \phi)$ and $\forall x \forall^1 y \phi \dashv\vdash \forall^1 y \forall x \phi$.

Recall from [31] that every **D**-formula $\phi(\mathbf{z})$ is logically equivalent to a formula of the form

$$\forall \mathbf{x} \exists \mathbf{y} \left(\bigwedge_{i \in I} =(\mathbf{x}_i, \mathbf{y}_i) \wedge \alpha(\mathbf{x}, \mathbf{y}, \mathbf{z}) \right), \quad (4)$$

where each $\mathbf{x}_i$ are from $\mathbf{x}$, each $\mathbf{y}_i$ is from $\mathbf{y}$, and α is first-order. In the next theorem we derive a similar normal form for formulas in **D** $\oplus$ **FOT**[↓]. See Appendix for the proof.

Theorem 17. Every **D** $\oplus$ **FOT**[↓]-formula $\phi(\mathbf{z})$ is semantically equivalent to, and provably implies a formula of the form

$$\forall^1 \mathbf{v} \forall \mathbf{x} \exists \mathbf{y} \left(\bigwedge_{i \in I} =(\mathbf{x}_i, \mathbf{y}_i) \wedge \alpha(\mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \right), \quad (5)$$

where each $\mathbf{x}_i$ are from $\mathbf{x}$, each $\mathbf{y}_i$ is from $\mathbf{y}$, and α is first-order.

Recall also from [27] that for every **D**-sentence ψ in normal form (4), there is a first-order sentence Ψ of infinite length (called the *game expression* of ψ) such that for any countable model M , $M \models \psi$ iff $M \models \Psi$. Moreover, the infinitary first-order sentence Ψ can be approximated by some first-order sentences Ψ_n ($n \in \mathbb{N}$) of finite length in the sense that for any recursively saturated (or finite) model M , $M \models \Psi$ iff $M \models \Psi_n$ for all $n \in \mathbb{N}$. Also, in the system of **D** one derives $\psi \vdash_{\mathbf{D}} \Psi_n$ for any $n \in \mathbb{N}$. Now, for any $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ -sentence $\phi = \forall^1 v \psi$ of the form (5) with ψ a **D**-sentence, it is not hard to show (by the same argument as in [27]) that the game expression Φ^* of ϕ can be defined as $\forall v \Psi$, and the n -approximation Φ_n^* can be defined as $\forall v \Psi_n$ for each $n \in \mathbb{N}$. Next, as in [27], we show that every n -approximation Φ_n^* can be derived from the formula ϕ .

Theorem 18. *For any $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ -sentence ϕ and any $n \in \mathbb{N}$, $\phi \vdash \Phi_n^*$.*

Proof. W.l.o.g. we may assume that the $\mathcal{L}$ -sentence $\phi = \forall^1 v \psi(v)$ is of the form (5) with $\psi(v)$ a **D**-sentence. Let c be a sequence of new constant symbols. Observe that the $\mathcal{L}(c)$ -sentence $\psi(c/v)$ is in the normal form (4) for **D**-formulas. By the result in [27] we have $\psi(c/v) \vdash \Psi_n(c/v)$. Thus, by Proposition 8(i), $\forall^1 I$ and $\forall^1 \forall \text{Trs}$, we derive $\forall^1 v \psi(v) \vdash \psi(c/v) \vdash \Psi_n(c/v) \vdash \forall^1 v \Psi_n(v) \vdash \forall v \Psi_n(v)$, thereby $\phi \vdash \Phi_n^*$. $\square$

Theorem 19 (Completeness). *For any set Γ of $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$ -sentences and an $\mathbf{FOT}^\downarrow$ -sentence θ , $\Gamma \models \theta \implies \Gamma \vdash \theta$.*

Proof. We only provide a sketch of the proof, which combines the arguments in [27] and in [33]. Suppose $\Gamma \not\models \theta$. Then $\Gamma, \sim \theta \not\models \perp$ by RAA, where $\sim \theta = \neg \theta$ as θ is an $\mathbf{FOT}^\downarrow$ -sentence. Let Γ^* be the set of all approximations of sentences in $\Gamma \cup \{\neg \theta\}$. By Theorem 18, we have $\Gamma^* \not\models \perp$. Since restricted to first-order formulas our extended system (or the system of **D** as defined in [27]) has the same rules as the deduction system of the usual first-order logic, we derive $\Gamma^* \not\models_{\mathbf{FO}} \perp$. From this point on we follow exactly the argument in [27] to find a model M for $\Gamma \cup \{\neg \theta\}$. Thus, $M \models \Gamma$ and $M \not\models \theta$. $\square$

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Appendix

Proof of Proposition 9

Item (i): The right to left direction follows from $\subseteq \text{Pro}$ (applied to repeated arguments in the inclusion atom) and rules of identity. For the left to right direction, $xy\xi \subseteq vv\eta \vdash x\xi \subseteq v\eta$ follows from $\subseteq \text{Pro}$. Next, by $\subseteq \text{Pro}$, rules of identity and $\subseteq \text{Cmp}$ we have $xy\xi \subseteq vv\eta \vdash xy \subseteq vv \vdash xy \subseteq vv \wedge v = v \vdash x = y$.

Item (ii): For the right to left direction, by Proposition 8(iii)(iv), it suffices to show the contrapositive $\sim \xi \subseteq \eta \vdash \exists^1 x (x \subseteq \xi \wedge \sim x \subseteq \eta)$. For any sequence c of fresh constant symbols, we have $c \subseteq \xi, \sim c \subseteq \eta \vdash \exists^1 x (x \subseteq \xi \wedge \sim x \subseteq \eta)$ by Proposition 8(i). Then the desired clause follows from $\sim \subseteq E$. For the other direction, by $\forall^1 I$, it suffices to show that $\xi \subseteq \eta \vdash c \subseteq \xi \rightarrow c \subseteq \eta$ for c a sequence of fresh constant symbols, which is further reduced to showing that $\xi \subseteq \eta, c \subseteq \xi \vdash c \subseteq \eta$. But this follows from $\subseteq \text{Tr}$.

Item (iii): We first show $\text{con}(z) \vdash w\xi \subseteq z\eta \rightarrow (w = z \wedge \xi \subseteq \eta)$, which is equivalent to $\text{con}(z), w\xi \subseteq z\eta \vdash w = z \wedge \xi \subseteq \eta$. By $\subseteq W_{\text{con}}$ we have $\text{con}(z), w \subseteq z \vdash wz \subseteq zz$. By item (i), $wz \subseteq zz \vdash w = z$. Hence, by $\subseteq \text{Pro}$ the desired clause follows. Next, we show $\text{con}(z), w = z, \xi \subseteq \eta \vdash w\xi \subseteq z\eta$. Again by $\subseteq W_{\text{con}}$ we have that $\xi \subseteq \eta, \text{con}(z) \vdash z\xi \subseteq z\eta$, and thus the desired clause follows from rules of identity.

Item (iv): The direction $\text{con}(x), x \subseteq v \vdash \exists^1 y (xy \subseteq vu)$ is given by $\subseteq W_{\exists^1 I}$, and the other direction $\text{con}(x), \exists^1 y (xy \subseteq vu) \vdash x \subseteq v$ follows easily from $\subseteq \text{Pro}$.

Item (v): We first show the left to right direction, which, by $\forall^1 I$, is reduced to showing that $\lambda(z) \vdash c \subseteq z \rightarrow \lambda(c)$ for c a sequence of fresh constant symbols. But this follows from $\subseteq \text{Cmp}$. Next, we show the other direction, which is equivalent to the contrapositive $\sim \lambda(z) \vdash \exists^1 w (w \subseteq z \wedge \sim \lambda(w))$. For any sequence c of fresh constant symbols, we have $c \subseteq z, \sim \lambda(c) \vdash \exists^1 w (w \subseteq z \wedge \sim \lambda(w))$ by Proposition 8(i). Then the desired clause follows from $\sim \lambda E$. $\square$

Proof of Lemma 12

We prove that $\Delta \vdash_{\text{FO}} \delta$ implies $\Delta^*(c/x) \vdash_{\text{FOT}} \delta^*(c/x)$ by induction on the depth of the proof tree of $\Delta \vdash_{\text{FO}} \delta$. If the proof tree has depth 1, then either $\delta \in \Delta$ or δ is the identity axiom $t = t$. In both cases $\Delta^*(c/x) \vdash_{\text{FOT}} \delta^*(c/x)$ trivially follows in our system.

If the proof tree has depth > 1 , and the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of a rule for $\neg$ or $\wedge$ or $\vee$ in **FO**, then we derive $\Delta^*(c/x) \vdash_{\mathbf{FOT}} \delta^*(c/x)$ by applying the induction hypothesis and the corresponding (classical) rules for $\sim$ or $\wedge$ or $\vee$ in our system for **FOT**.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\exists I$ rule:

$$\frac{\begin{array}{c} \vdots \pi \\ \Delta \vdash \alpha(t/x) \end{array}}{\Delta \vdash \exists x \alpha} \exists I$$

where the variables and constant symbols occurring in the term t are, respectively, among vy and d (denoted as $t(vy, d)$), then the corresponding derivation in **FOT** is:

$$\frac{\begin{array}{c} \vdots \pi^* \\ \Delta^*(c/y) \vdash \alpha^*(t(c'/v, c/y)/x, c/y) \end{array} \quad \frac{\frac{\Delta^*(c/y) \vdash \text{con}(c'cd)}{\Delta^*(c/y) \vdash \text{con}(c'cd)} \text{conl}}{\Delta^*(c/y) \vdash \text{con}(c'cd)} \text{conl}}{\Delta^*(c/y) \vdash \exists^1 x \alpha^*(x, c/y)} \exists^1 I$$

where π^* is a derivation corresponding to π given by the induction hypothesis.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\exists E$ rule:

$$\frac{\begin{array}{c} \vdots \pi_1 \\ \Delta \vdash \exists x \alpha \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ \Delta, \alpha(v/x) \vdash \delta \end{array}}{\Delta \vdash \delta} \exists E$$

where $v \notin Fv(\Delta \cup \{\alpha, \delta\})$, then the corresponding derivation in **FOT** is:

$$\frac{\begin{array}{c} \vdots \pi_1^* \\ \Delta^*(c/y) \vdash \exists^1 x \alpha^*(x, c/y, c'/u) \end{array} \quad \begin{array}{c} \vdots \pi_2^* \\ \Delta^*(c/y), (\alpha(v/x))^*(d/v, c/y, c'/u) \vdash \delta^*(c/y) \end{array}}{\Delta^*(c/y) \vdash \delta^*(c/y)} \exists^1 E$$

where π_1^*, π_2^* are, respectively, derivations corresponding to π_1, π_2 given by the induction hypothesis, and d is a fresh constant symbol.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\forall I$ rule:

$$\frac{\begin{array}{c} \vdots \pi \\ \Delta \vdash \alpha(v/x) \end{array}}{\Delta \vdash \forall x \alpha} \forall I$$

where $v \notin Fv(\Delta \cup \{\alpha\})$, then the corresponding derivation in **FOT** is:

$$\frac{\begin{array}{c} \vdots \pi^* \\ \Delta^*(c/y) \vdash (\alpha(v/x))^*(d/v, c/y) \end{array}}{\Delta^*(c/y) \vdash \forall^1 x \alpha^*(x, c/y)} \forall^1 I$$

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\forall E$ rule:

$$\frac{\begin{array}{c} \vdots \pi \\ \Delta \vdash \forall x \alpha \end{array}}{\Delta \vdash \alpha(t/x)} \forall E$$

where $t = t(vy, d)$, then the corresponding derivation in **FOT** is:

$$\frac{\begin{array}{c} \vdots \pi^* \\ \Delta^*(c/y) \vdash \forall^1 x \alpha^*(x, c/y) \end{array} \quad \frac{\frac{\Delta^*(c/y) \vdash \text{con}(c'cd)}{\Delta^*(c/y) \vdash \text{con}(c'cd)} \text{conl}}{\Delta^*(c/y) \vdash \text{con}(c'cd)} \text{conl}}{\Delta^*(c/y) \vdash \alpha^*(t(c'/v, c/y)/x, c/y)} \forall^1 E$$

□

Proof of Theorem 15 (Soundness Theorem of $\mathbf{D} \oplus \mathbf{FOT}^\downarrow$)

We only verify the soundness of $\forall^1 \text{Ext}$, by showing that $\forall^1 x \phi \vee \psi \equiv \exists y z \forall^1 x ((y = z \wedge \phi) \vee (y \neq z \wedge \psi))$. For the left to right direction, suppose $M \models_X \forall^1 x \phi \vee \psi$, and we may w.l.o.g. also assume that $x, y, z \notin \text{dom}(X)$. Then there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $M \models_Y \forall^1 x \phi$ and $M \models_Z \psi$. Let a, b be two distinct elements in M . Define $F : X \rightarrow \wp(M) \setminus \{\emptyset\}$ as $F(s) = \{a\}$, and define $G : X(F/y) \rightarrow M$ by taking

$$a \in G(s) \iff s \upharpoonright \text{dom}(X) \in Y, \quad \text{and} \quad b \in G(s) \iff s \upharpoonright \text{dom}(X) \in Z.$$

Putting $X' = X(F/y)(G/z)$ we show that $M \models_{X'(c/x)} (y = z \wedge \phi) \vee (y \neq z \wedge \psi)$ for arbitrary $c \in M$. Define $Y' = \{s \in X'(c/x) \mid s(z) = a\}$ and $Z' = X'(c/x) \setminus Y'$. Clearly, $Y' \cup Z' = X'(c/x)$, $M \models_{Y'} y = z$ and $M \models_{Z'} y \neq z$. Since $M \models_Z \psi$ and $x, y, z \notin \text{dom}(Z)$, we have $M \models_{Z'} \psi$. Also, since $M \models_Y \forall x \phi$, we have $M \models_{Y(c/x)} \phi$, which implies $M \models_{Y'} \phi$.

For the right to left direction, suppose $M \models_X \exists y z \forall^1 x ((y = z \wedge \phi) \vee (y \neq z \wedge \psi))$. Then there exist appropriate functions F, G s.t. for any $a \in M$, there exists $Y_a \subseteq X(F/y)(G/z)(a/x) = X'(a/x)$ s.t. $M \models_{Y_a} y = z \wedge \phi$ and $M \models_{X'(a/x) \setminus Y_a} y \neq z \wedge \psi$.

Claim: For any $a, b \in M$, $Y_a \upharpoonright \text{dom}(X) = Y_b \upharpoonright \text{dom}(X)$. Indeed, for any $s \in Y_a \subseteq X'(a/x)$, we have $s(y) = s(z)$. For $s' = s(b/x) \in X'(b/x)$, we must also have $s'(y) = s'(z)$, thus $s' \in Y_b$. Hence, $s \upharpoonright \text{dom}(X) = s' \upharpoonright \text{dom}(X) \in Y_b \upharpoonright \text{dom}(X)$. This shows that $Y_a \upharpoonright \text{dom}(X) \subseteq Y_b \upharpoonright \text{dom}(X)$. The other inclusion is proved similarly.

Now, to show $M \models_X \forall^1 x \phi \vee \psi$, let $Y = Y_a \upharpoonright \text{dom}(X)$ and $Z = X \setminus Y$ for any $a \in M$. Since $M \models_{X'(a/x) \setminus Y_a} \psi$ and $Z = X \setminus (Y_a \upharpoonright \text{dom}(X)) = (X'(a/x) \setminus Y_a) \upharpoonright \text{dom}(X)$, we obtain $M \models_Z \psi$. Meanwhile, for any $b \in M$, by the claim, $Y = Y_b \upharpoonright \text{dom}(X)$. Since $M \models_{Y_b} \phi$ and $Y_b \upharpoonright (\text{dom}(X) \cup \{x\}) = Y(b/x)$, we obtain $M \models_{Y(b/x)} \phi$. $\square$

Proof of Proposition 16

Item (i): By rules of identity we have $\vdash c = c \wedge z = z$, which implies $\vdash \exists x \forall y (x = c \wedge z = z)$. Now, by $=(\cdot)I$ we derive $\vdash \forall y \exists x (=z, x) \wedge x = c \wedge z = z$, which yields $\vdash (=z, c)$.

Item (ii): For the direction $=(cx, y) \vdash =(x, y)$, by applying $=(\cdot)wE$ we derive that $=(cx, y), =(c), \text{con}(x) \vdash =(y)$. Since $\vdash =(c)$ by item (i), we conclude by $=(\cdot)wI$ that $=(cx, y) \vdash =(x, y)$. The other direction is derived similarly by applying $=(\cdot)wE, =(\cdot)wI$ and item (i).

Items (iii): By Proposition 8(i) and $\forall^1 I$ it suffices to show $Qu(\phi(c/v) \wedge =(cx, y)) \dashv\vdash Qu(\phi(c/v) \wedge =(x, y))$ and $Qu(\phi(c/v) \wedge =(x, c)) \dashv\vdash Qu(\phi(c/v))$. But these follow from items (i) and (ii).

Item (iv): The direction $\exists x (=x) \wedge \phi \vdash \exists^1 x \phi$ follows easily from $\exists E$ and $\exists^1 I$. For the other direction, by Proposition 16(i), we have $\exists^1 x \phi \vdash \phi(c/x)$. Moreover, by rules of identity, $\vdash =(c)$ and $\exists I$, we have $\phi(c/x) \vdash \exists x (x = c \wedge =(c) \wedge \phi(c/x)) \vdash \exists x (=x) \wedge \phi$. Putting these together we obtain $\exists^1 x \phi \vdash \exists x (=x) \wedge \phi$.

Item (v): We first prove the right to left direction. By $\exists E$ it suffices to prove that $=(x), =(y), (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \vee \psi$. We first derive $=(x), =(y) \vdash x = y \vee x \neq y$ by $\vee wI$. Next, by applying $\vee \text{Sub}, \perp \vee E$ and $\vee I$, we derive

$$x = y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \vee (x = y \wedge x \neq y) \vdash \phi \vee \perp \vdash \phi \vee \psi$$

and similarly $x \neq y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \vee \psi$. Hence, we conclude by $\vee E$ that $x = y \vee x \neq y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \vee \psi$, from which the desired clause follows.

For the left to right direction, by $\forall I$, it suffices to prove that the right formula is derivable from both ϕ and ψ . We now first derive the right hand side from ϕ . By the rules of identity, $\vdash \exists x \forall z (x = x)$ for some fresh variables x, z . Thus, we conclude by applying $=(\cdot)I$ that $\vdash \forall z \exists x (=x \wedge x = x)$, which reduces to $\vdash \exists x =x$. Next, we derive by rules of identity that $\vdash \exists x (=x \wedge \exists y (x = y))$ and thus $\vdash \exists x \exists y (=x \wedge =y \wedge x = y)$. Lastly, we conclude by the introduction rule of $\vee$ that

$$\phi \vdash \exists x \exists y (=x \wedge =y) \wedge x = y \wedge \phi \vdash \exists x \exists y (=x \wedge =y) \wedge ((x = y \wedge \phi) \vee (x \neq y \wedge \psi)).$$

Similarly, to derive the right hand side from ψ , first note that by Dom we have $\vdash \exists x \exists y (x \neq y)$ for some fresh variables x, y , which then yields $\vdash \exists x \forall z \exists y (x \neq y)$. Then, by a similar argument as above, we derive by applying $=(\cdot)I$ that $\vdash \exists x (=x) \wedge \exists y (x \neq y)$, and that $\vdash \exists y (=y) \wedge \exists x (=x \wedge (x \neq y))$, from which the required clause follows.

Item (vi): $\forall x \forall^1 y \phi \dashv \vdash \forall^1 y \forall x \phi$ follows easily from Proposition 8(i). For the other clause, by $\exists E$, it suffices to prove $\forall^1 y \phi(x, y, z) \vdash \forall^1 y \exists x (=x \wedge \phi)$. We derive by Proposition 8(i) and the rules for $\forall, \exists$ that $\forall^1 y \phi(x, y, z) \vdash \phi(x, c, z) \vdash \exists x \forall w \phi(x, c, z)$ for some fresh constant symbol c and variable w . Moreover, by $=(\cdot)I$ we derive that $\exists x \forall w \phi(x, c, z) \vdash \forall w \exists x (=x \wedge \phi(x, c, z)) \vdash \exists x (=x \wedge \phi(x, c, z))$. Putting these together and by applying $\forall^1 I$ we conclude that $\forall^1 y \phi(x, y, z) \vdash \exists x (=x \wedge \phi(x, c, z)) \vdash \forall^1 y \exists x (=x \wedge \phi)$. $\square$

Proof of Theorem 17

We adapt the argument for the normal form proof in [27]. We give the semantic and syntactic proof at the same time, and the semantic equivalence clearly follows from the syntactic equivalence (by soundness theorem) whenever the latter is available in the following steps of the proof. First, rewrite every occurrence of $\theta \vee \chi$ and $\exists^1 x \eta$ in ϕ using the equivalent formulas with logical constants from **D** given by Proposition 16(iv)(v), and denote the resulting provably equivalent formula (which is $\vee$ and $\exists^1$ -free) by ϕ' . Next, turn the formula ϕ' into a formula $Q_1 x \dots Q_n \theta$ in prenex normal form, where each $Q_i \in \{\forall, \exists, \forall^1\}$ and θ is quantifier-free. This step is done, as in [27], by induction on the complexity of the formula ϕ' , where the inductive steps with $\forall^1$ follow from the provable equivalences given by Proposition 8(ii) and the rule $\forall^1 \text{Ext}$.

Now, since $\theta(x)$ is a formula of **D**, we proceed in the same way as in [27] to turn θ into a formula of the form $\forall y \exists z (\bigwedge_{i \in I} d_i(x, y, z) \wedge \alpha(x, y, z)) = \forall y \exists z \theta'$, where each d_i is a dependence atom and α is first-order. The formula θ is semantically equivalent to $\forall y \exists z \theta'$, and in the deduction system of **D** we can prove (as is done in [27]) that $\theta \vdash \forall y \exists z \theta'$. Thus, altogether we now have $\phi \equiv Qx \forall y \exists z \theta'$ and $\phi \vdash Qx \forall y \exists z \theta'$.

To turn the formula $Qx \forall y \exists z \theta'$ finally into the required normal form (5), we swap the order of the quantifiers using an inductive argument similar to that in [27], where the inductive step for $\forall^1$ is taken care of in the deduction system by applying Proposition 16(vi), and on the semantic side by using the equivalences $\exists x \forall^1 y \psi(x, y, z) \equiv \forall^1 y \exists x (=x \wedge \psi)$ and $\forall x \forall^1 y \psi \equiv \forall^1 y \forall x \psi$ (we leave it for the reader to verify). To conclude the proof, we apply Proposition 16(iii) to remove variables quantified by $\forall^1$ in dependence atoms $d_i(x, y, z)$ in the first conjunct of the quantifier-free formula. $\square$