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A VERSION OF STOKES' THEOREM USING TEST CURVES

VENDULA HONZLOVÁ EXNEROVÁ, JAN MALÝ, AND OLLI MARTIO

ABSTRACT. We prove that a parametric Lipschitz surface of codimension 1 in a smooth manifold induces a boundary in the sense of currents (roughly speaking, surrounds a “domain” with an eventual multiplicity and together with it forms a pair for the Stokes theorem) if and only if it passes a test in terms of crossing the surface by “almost all” curves. We use the AM -modulus recently introduced in [52] to measure the exceptional family of curves.

1. INTRODUCTION

Let $\mathcal{G}$ be a smooth n -dimensional manifold, $\mathcal{O}$ be a relatively compact open subset of $\mathcal{G}$ with a smooth boundary $\partial\mathcal{O}$ and ω a smooth differential $(n-1)$ -form on the closure of $\mathcal{O}$. Then the Stokes theorem

$$\int_{\partial\mathcal{O}} \omega = \int_{\mathcal{O}} d\omega$$

holds. In the Euclidean setting, where $\mathcal{G} = \mathbb{R}^n$, this reduces to the divergence (Gauss-Green) theorem

$$\int_{\partial\mathcal{O}} f \cdot \mathbf{n} d\mathcal{H}^{n-1} = \int_{\mathcal{O}} \operatorname{div} f dx,$$

for $f \in \mathcal{C}^1(\overline{\mathcal{O}}, \mathbb{R}^n)$. Here $\mathbf{n}$ denotes the exterior normal to $\mathcal{O}$.

We can relax the requirements on $\mathcal{O}$. In the Euclidean setting, assume that $\mathcal{O} \subset \mathbb{R}^n$ is relatively compact and BV (alias of finite perimeter), so that the “boundary” $\partial^*\mathcal{O}$ (precisely, the so called reduced boundary) is only $(n-1)$ -rectifiable. Then the divergence formula still holds, see [32, Theorem 5.8.2], it is based on the theory of De Giorgi and Federer.

The regularity of f can be much weakened. Particularly deep results have been obtained for divergence measure vector fields, see e.g. Chen, Torres and Ziemer [7], Ziemer [31] or Šilhavý [27, 28, 29]. Another direction of generalization is towards vector field whose divergence is not Lebesgue integrable, but still represented by a function. This leads to non-absolutely convergent integration. For results in this spirit see papers and books by Pfeffer, e.g. [24]. For our goals, the regularity of f is not important.

Through coordinate maps, we can easily formulate what we mean by BV regularity also in the manifold setting. We refer to a series of paper [15], [47], [48] by Krickeberg for a treatment of Stokes theorem on manifolds with historical references.

The full generality of the Stokes theorem with rough boundaries is thoroughly studied in [40]; this leads to the theory of currents. We do not use the theory of currents, however, its language allows us to speak on our results in context of the developed theory.

Key words and phrases. Stokes theorem; divergence theorem; modulus of path family; functions of bounded variation.

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We are interested in the situation that $\mathcal{O}$ is slightly generalized to a compactly supported multiplicity function u , so that our object replacing $\mathcal{O}$ is a full-dimensional integral current Ξ on $\mathcal{G}$ and $\partial\mathcal{O}$ is replaced by $\partial\Xi$, which means the boundary in the sense of currents. If the multiplicity function has bounded variation, then the current $\partial\Xi$ is rectifiable and thus it can be represented by integration over a Lipschitz parametric surface [40, Sections 3.2, 4.1].

Our aim is to study the converse:

Given a parametric $(n-1)$ dimensional surface φ into $\mathcal{G}$, how to recognize that it represents a boundary of an n -dimensional integral current on $\mathcal{G}$?

If φ is a rectifiable curve in $\mathbb{R}^2$, then it represents a boundary of a current if and only if it is closed (Green's theorem). The higher dimensional counterpart of “closed curve” is difficult to describe. The surface φ should represent a boundaryless current, but this property is not easy to verify.

Additional complications appear in the setting of manifold, namely, it can happen that a closed curve does not represent any boundary (see Example 6).

For simplicity, we describe our results first in the flat case $\mathcal{G} = \Omega$ where $\Omega \subset \mathbb{R}^n$ is an open set. In Section 7 we explain how to transfer them into the true manifold setting. Throughout the paper, measure is always a Borel regular measure.

Definition 1. We define a *parametric surface* as a Lipschitz mapping $\varphi: E \rightarrow \Omega$, where $E \subset \mathbb{R}^{n-1}$ is a measurable set with $0 < |E| < \infty$. Then φ induces a vector valued measure $\vec{\nu}_\varphi$ which measures each Borel set $S \subset \Omega$ as

$$\vec{\nu}_\varphi(S) := \int_{E \cap \varphi^{-1}(S)} N_\varphi(y) dy,$$

where

$$N_\varphi = D_1\varphi \times \cdots \times D_{n-1}\varphi.$$

(For other representations and more details see 3.1 below.)

Note that we do not assume any connectedness of the domain E , so that the surface can be glued from fragments and, in fact, any $(n-1)$ -dimensional rectifiable current can be obtained as some $\vec{\nu}_\varphi$. Now, our main question can be interpreted as to find a criterion for the validity of the divergence theorem

$$(1) \quad \int_{\varphi(E)} f \cdot d\vec{\nu}_\varphi = \int_{\Omega} u \operatorname{div} f \, dx, \quad f \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n),$$

where u is an integer-valued “multiplicity function”.

We obtain the “usual” Gauss-Green theorem for sets when the multiplicity u attains just values 0 and 1. However, this may be false even if φ is one-to-one. We include several examples at the end of Section 6.

Note also that the multiplicity $u(x)$ is just the index of the point x with respect to a curve known from the elementary complex analysis if $n = 2$ and φ is a closed rectifiable curve.

Our criterion is based on *crossing number* of pairs (φ, γ) , where γ are used as test curves. Let $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ be a rectifiable curve of length ℓ and $(y, t) \in E \times [0, \ell]$. Then (y, t) is a couple of crossing if $\gamma(t) = \varphi(y)$. The crossing number $\otimes(\varphi, \gamma)$ is just the signed multiplicity of the set of couples of crossing. Each couple of crossing (y, t) is added with its sign; if $\gamma'(t)$ exists it is just

$$(2) \quad \det(\gamma'(t), D_1\varphi(y), \dots, D_{n-1}\varphi(y)) = \gamma'(t) \cdot N_\varphi(y);$$

however, we can determine the sign in much more general situations. It can happen that the crossing number does not make sense, for example, the number of couples of crossing can be infinite or the derivatives in (2) do not exist. Fortunately, these

bad curves can be neglected. For this purpose, we need to have tools to measure smallness of families of curves.

We use *approximation modulus*, or shortly *AM-modulus*, of a family of curves to measure its size. The concept of modulus of curve family goes back to Ahlfors and Beurling [34], has been developed by Fuglede [11] and thoroughly exploited in geometric function theory, see [54] for an overview. However, its approximation version is recent, it has been introduced in [52], [53], see also [42]. Both the *AM*- and the *M*₁-modulus employ *L*¹-functions but from a different point of view, see Ambrosio Gigli and Savaré [6], Ambrosio and Di Marino [3], Ambrosio, Di Marino and Savaré [4] for another closely related approach.

We refer to later sections for more precise definitions of notions like path, crossing number or approximation modulus. Using them, we can formulate here the main result of our paper, for simplicity in the Euclidean setting. We say that a property holds for *AM-a.e. curve* if the family of all curves for which the property fails has zero *AM*-modulus.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\varphi : E \rightarrow \Omega$ be a Lipschitz surface. The following statements are equivalent:*

- (i) *There exists a locally BV function $u : \Omega \rightarrow \mathbb{R}$ such that $Du = -\vec{\nu}_\varphi$.*
- (ii) *There exists a locally BV function $u : \Omega \rightarrow \mathbb{R}$ such that $u(x) \in \mathbb{N}$ for a.e. x and*

$$u(\gamma(\ell)) - u(\gamma(0)) = -\otimes(\varphi, \gamma)$$

for AM-a.e. path $\gamma : [0, \ell] \rightarrow \Omega$.

- (iii) *$\otimes(\varphi, \gamma) = 0$ for AM-a.e. closed path $\gamma : [0, \ell] \rightarrow \Omega$.*

Let us note that our criterion reminds testing by parallel lines. In fact, the first definitions of *BV* function were based on parallel lines, this goes back to Tonelli, Adams, Clarkson, Cesari, Levi and others, for a historical treatment we refer to [35, 3.12]. In connection with the Green theorem with multiplicity, a crossing criterion to calculate the index of a point with respect to a curve has been established by Král and Mařík [45]. A generalization to higher dimension has been investigated by Černý [26]. However, these criterions did not reach the ultimate generality and were also based on parallel lines. The notion of *AM*-modulus enables us to formulate criterions using curves. This is naturally required in the manifold setting where parallel lines have no meaning.

The paper is organized as follows: In Section 2 we fix the notation which is used throughout the paper. Section 3 analyses Lipschitz surfaces and their exceptional points which we want to omit.

Section 4 is devoted to crossing of curves through the Lipschitz surface. We show that *AM*-a.e. curve has good crossing properties and thus the aforementioned crossing number is well-defined.

In Section 5 we connect the crossing behavior of curves to a *BV* function whose derivative is concentrated on the Lipschitz surface.

Theorem 1 is then proved in Section 6 and the manifold case is handled in Sections 7 and 8.

2. PRELIMINARIES

2.1. Paths and moduli. Let X be a metric space. A continuous mapping $\gamma : [a, b] \rightarrow X$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non-zero total length; in this case γ can be reparametrized by its arclength. We use the convention that any path in consideration is already parametrized by its arclength (if not stated otherwise). We consider paths with various lengths but (with some abuse of notation) do not mark the dependence of the total length ℓ on γ . The

locus of γ is defined as $\gamma([0, \ell])$ and denoted by $\langle \gamma \rangle$. A closed path $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ is called *smooth* if its ℓ -periodic extension $\tilde{\gamma}: \mathbb{R} \rightarrow \mathbb{R}^n$ is continuously differentiable.

If $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ is a path, we consider two kinds of curvilinear integrals along γ . The integral of a scalar function $u: \langle \gamma \rangle \rightarrow \mathbb{R}$ over γ is defined as

$$\int_{\gamma} u \, ds = \int_0^{\ell} u(\gamma(t)) |\gamma'(t)| \, dt$$

(if the integral on the right makes sense), which simplifies to

$$\int_{\gamma} u \, ds = \int_0^{\ell} u(\gamma(t)) \, dt$$

if γ is parametrized by arclength. The vector valued measure τ_{γ} induced by γ acts on vector valued functions $f \in \mathcal{C}(\langle \gamma \rangle; \mathbb{R}^n)$ as

$$\int_{\langle \gamma \rangle} f \, d\tau_{\gamma} = \int_0^{\ell} f(\gamma(t)) \cdot \gamma'(t) \, dt.$$

Let Γ be a family of paths in $\mathbb{R}^n$. A Borel measurable function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$ is said to be *admissible* for Γ if $\int_{\gamma} \rho \, ds \geq 1$ for every $\gamma \in \Gamma$. Given $p \in [1, \infty)$, we define the M_p -modulus of Γ as

$$M_p(\Gamma) := \inf \left\{ \int_{\mathbb{R}^n} \rho^p \, dx : \rho \text{ is admissible for } \Gamma \right\}.$$

A property is said to hold for M_p -almost every curve if the M_p -modulus of the family of curves for which the property fails is zero.

The M_1 -modulus cooperates well with the theory of Sobolev spaces $W^{1,1}$, but not so well with the theory of BV spaces. For the purpose of investigation of BV functions on metric spaces, the AM -modulus introduced in [52] is more efficient.

Let Γ be a family of paths in $\mathbb{R}^n$. A sequence (ρ_k) of Borel measurable functions on $\mathbb{R}^n$ with values in $[0, \infty]$ is said to be *AM-admissible* (or simply *admissible*) for Γ if

$$\liminf_{k \rightarrow \infty} \int_{\gamma} \rho_k \, ds \geq 1, \quad \gamma \in \Gamma.$$

We define the AM -modulus of Γ as

$$AM(\Gamma) := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \rho_k \, dx : (\rho_k) \text{ is admissible for } \Gamma \right\}.$$

Note that the AM -modulus is countably subadditive, see [52, Theorem 3.1]. We also use the concept AM -a.e. curve as in the case of the M_p -modulus.

The analysis of behavior of a.e. paths generalizes the consideration of a.e. lines parallel to coordinate axes. We let Π_i denote the coordinate projection to the coordinate plane $\mathbb{H}_i := \{x \in \mathbb{R}^n : x_i = 0\}$, $i \in \{1, 2, \dots, n\}$. For a set $A \subset \mathbb{R}^n$ we let $|A|$ denote its Lebesgue measure. Recall that the path γ_0 below is parametrized by arc length.

Proposition 1. *Let Γ be a family of paths in $\mathbb{R}^n$, $AM(\Gamma) = 0$. Let $\gamma^0: [0, \ell] \rightarrow \mathbb{R}^n$ be a fixed C^1 -smooth path and γ^x its translate, i.e. $\gamma^x(t) = x + \gamma^0(t)$, $t \in [0, \ell]$. If A is the set of all points x such that there exists a subpath $\tilde{\gamma}^x \in \Gamma$ of γ^x , then $|A| = 0$.*

In particular, if $i \in \{1, 2, \dots, n\}$, then for a.e. $y \in \mathbb{H}_i$, the line $\Pi_i^{-1}(y)$ does not contain any segment belonging to Γ .

Proof. By [42, Theorem 7], there is a sequence (ρ_k) of Borel functions on $\mathbb{R}^n$ such that

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \rho_k \, dx \leq 1 \text{ but } \lim_{k \rightarrow \infty} \int_{\gamma} \rho_k \, ds = \infty$$

for each $\gamma \in \Gamma$. It follows that

$$\lim_{k \rightarrow \infty} \int_{\gamma^x} \rho_k ds = \infty$$

for each $x \in A$. We may assume that the functions ρ_k are lower semicontinuous. Then the functions

$$x \mapsto \int_{\gamma^x} \rho_k ds$$

are lower semicontinuous because by the Fatou lemma

$$\begin{aligned} \liminf_{y \rightarrow x} \int_{\gamma^y} \rho_k ds &= \liminf_{y \rightarrow x} \int_0^\ell \rho_k(\gamma^y(t)) dt \geq \int_0^\ell \liminf_{y \rightarrow x} \rho_k(\gamma^y(t)) dt \\ &\geq \int_0^\ell \rho_k(\gamma^x(t)) dt = \int_{\gamma^x} \rho_k ds \end{aligned}$$

and hence the set

$$A_{p,k} := \left\{ x : \int_{\gamma^x} \rho_k ds > p \right\}$$

is open for each $k, p \in \mathbb{N}$.

Assume first that $\gamma^0 = (\gamma_1, \dots, \gamma_n)$ with $0 < \alpha < \gamma'_n$ on $[0, \ell]$. Fix $\tau \in \mathbb{R}$. We estimate the measure of

$$A_{p,k}^\tau := \{ \hat{x} \in \mathbb{R}^{n-1} : (\hat{x}, \tau) \in A_{p,k} \}$$

and introduce a mapping

$$\Phi^\tau(\hat{x}, t) = \gamma^{(\hat{x}, \tau)}(t), \quad (\hat{x}, t) \in \mathbb{R}^{n-1} \times [0, \ell]$$

and observe that Φ^τ is one-to-one and its Jacobian satisfies

$$J\Phi^\tau(\hat{x}, t) = \gamma'_n(t) \geq \alpha.$$

Then

$$\begin{aligned} \alpha p |A_{p,k}^\tau| &\leq \alpha \int_{A_{p,k}^\tau} \left(\int_{\gamma^{(\hat{x}, \tau)}} \rho_k dt \right) d\hat{x} = \alpha \liminf_{k \rightarrow \infty} \int_{A_{p,k}^\tau \times [0, \ell]} \rho_k(\gamma^{(\hat{x}, \tau)}(t)) d\hat{x} dt \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_{p,k}^\tau \times [0, \ell]} \rho_k(\Phi^\tau(\hat{x}, t)) J\Phi^\tau(\hat{x}, t) d\hat{x} dt \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \rho_k(x) dx \leq 1. \end{aligned}$$

Using measurability of $A_{p,k}$, the Fubini theorem yields that

$$|A_{p,k} \cap (-R, R)^n| \leq \int_{-R}^R |A_{p,k}^\tau| d\tau \leq \frac{2R}{p\alpha}, \quad R > 0.$$

In the general case we find a partition

$$0 = t_0 < t_1 < \dots < t_m = \ell$$

such that at least one coordinate γ_i of γ is strictly monotone on each $[t_{j-1}, t_j]$ with $|\gamma'_i| \geq \alpha$ for some $\alpha > 0$. Let

$$A_{p,k,j} = \left\{ x \in \mathbb{R}^n : \int_{t_{j-1}}^{t_j} \rho_k(\gamma^x(t)) dt > \frac{p}{m} \right\}.$$

Then

$$A_{p,k} \subset \bigcup_{j=1}^m A_{p,k,j}$$

As above, we compute that

$$|A_{p,k,j} \cap (-R, R)^n| \leq \frac{2Rm}{p\alpha},$$

and hence

$$|A_{p,k} \cap (-R, R)^n| \leq \frac{2Rm^2}{p\alpha}, \quad R > 0.$$

Since

$$A \subset \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{k=q}^{\infty} A_{p,k},$$

we deduce that

$$|A \cap (-R, R)^n| = \inf_{p \in \mathbb{N}} \frac{2Rm^2}{p\alpha} = 0, \quad R > 0,$$

so that $|A| = 0$. $\square$

Remark 1. Proposition 1 holds for the M_p -modulus with even a simpler proof.

2.2. Functions and sets of bounded variation. We refer to [32], [35] and [39] for introduction to the theory of BV functions. Here, we recall some concepts needed in the sequel. For a set $E \subset \mathbb{R}^n$ and $d \geq 0$, $\mathcal{H}^d(E)$ denotes the standard normalized d -dimensional Hausdorff measure of E .

Let Ω be an open set in $\mathbb{R}^n$. We say that $u \in L^1(\Omega)$ is a function of *bounded variation* or simply BV in Ω if its distributional gradient Du can be represented as a vector-valued finite Radon measure Du in Ω . This means that

$$\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \varphi \cdot Du$$

for each function $\varphi \in C_c^1(\Omega, \mathbb{R}^n)$ and $Du = \sigma(x)|Du|$ where $|Du|$ is a finite Radon measure (the variation measure of Du), σ is a $|Du|$ measurable function with $|\sigma(x)| = 1$ $|Du|$ -a.e. in Ω and $\|Du\| = |Du|(\Omega) < \infty$ is the total variation of Du in Ω . A function $u \in L^1_{\text{loc}}(\Omega)$ is *locally BV* in Ω if $u|_U$ is a BV function in each $U \subset \subset \Omega$. The spaces of BV and locally BV functions in Ω are denoted as $BV(\Omega)$ and $BV_{\text{loc}}(\Omega)$, respectively.

If $u \in L^1_{\text{loc}}(\Omega)$, then a point $z \in \Omega$ is said to be a *Lebesgue point* for u if

$$\lim_{r \rightarrow 0+} \int_{B(z,r)} |u(x) - u(z)| \, dx = 0.$$

We define the L^1 -approximate discontinuity set S_u as the set of non-Lebesgue points for u . Given $z \in \mathbb{R}^n$, $r > 0$ and $\mathbf{n} \in \mathbb{S}^{n-1}$, we write

$$B_{\mathbf{n}}(z, r) = \{x \in B(z, r) : (x - z) \cdot \mathbf{n} > 0\}.$$

We say that $z \in \Omega$ is a L^1 -approximate jump point for u if there exist $a, b \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$(3) \quad \lim_{r \rightarrow 0+} \int_{B_{\mathbf{n}}(z,r)} |u(x) - b| \, dx = \lim_{r \rightarrow 0+} \int_{B_{-\mathbf{n}}(z,r)} |u(x) - a| \, dx = 0.$$

Define the *normal vector* of u at z by $\mathbf{n}_u(z) = (b - a)\mathbf{n}$ where $\mathbf{n}$ is given by the preceding formula; the vector $\mathbf{n}_u(z)$ is unique at L^1 -approximate jump points. We define the L^1 -approximate jump set J_u of u as the set of all L^1 -approximate jump points for u .

We say that $\bar{u} : \Omega \rightarrow \mathbb{R}$ is a *precise representative* of $u \in BV_{\text{loc}}(\Omega)$ if

$$\bar{u}(z) = \lim_{r \rightarrow 0+} \frac{1}{|B(z, r)|} \int_{B(z, r)} u(x) \, dx$$

whenever for z this limit exists and is finite. We do not require anything at the remaining points, which, however, occupy only a set of $\mathcal{H}^{n-1}$ -measure zero by [35,

Theorem 3.78] or [32, Theorem 5.9.6]. Indeed, the set where the precise representative remains undetermined is contained in $S_u \setminus J_u$. At a jump point z of u we observe $\bar{u}(z) = (a + b)/2$ where a and b are as in (3).

The following theorem, see [43, Theorem 6], connects the BV property to the AM -modulus and makes the constructions for Theorem 1 possible.

Theorem 2. *If u is a precise representative of a locally BV function in Ω , then $u \circ \gamma$ has bounded total variation on $[0, \ell]$ for AM -a.e. path γ in Ω .*

A measurable set $E \subset \mathbb{R}^n$ has *finite* or *locally finite perimeter* in Ω if the function χ_E belongs to $BV(\Omega)$ or to $BV_{loc}(\Omega)$, respectively. We let $DE = D\chi_E$ and if E has locally finite perimeter, then the *reduced boundary* ∂^*E of E is the set such that $\|DE\|(\mathbb{R}^n \setminus \partial^*E) = 0$, see [39, 5.7.].

The following classical result, see e.g. [33, Theorem 17.4.6] or [18, Chapter V, §5, Theorem 6], gives a criterion for the existence of a function whose derivative is a given vector field.

Proposition 2. *Suppose that $f \in \mathcal{C}(\Omega; \mathbb{R}^n)$ and*

$$\int_{\langle \gamma \rangle} f \, d\tau_\gamma = 0$$

for each smooth closed curve γ in Ω . Then there exists $u \in \mathcal{C}^1(\Omega)$ such that $\nabla u = f$ in Ω .

2.3. Degree theory. We recall some properties of the topological degree which are needed in the sequel, for these properties see e.g. [8] and [57].

Let $D \subset \mathbb{R}^n$ be a bounded domain (open connected set) and $f : \bar{D} \rightarrow \mathbb{R}^n$ a continuous mapping. Then the topological degree $\deg(y, f, D)$ of f is an integer valued function defined at every point $y \notin f(\partial D)$ and a constant in each component of $\mathbb{R}^n \setminus f(\partial D)$. Moreover,

- $\deg(y, f, D) = 0$ whenever $y \notin f(\bar{D})$,
- if f is a homeomorphism, then $\deg(y, f, D) = 1$ for $y \in f(D)$ and f is sense-preserving and $= -1$ when f is sense-reversing,
- If $H : [0, 1] \times \bar{D} \rightarrow \mathbb{R}^n$ is a homotopy and $y \notin H([0, 1] \times \partial D)$, then $t \mapsto \deg(y, H(t, \cdot), D)$ is constant (invariance under homotopy).

2.4. Lipschitz mappings. Let $E \subset \mathbb{R}^d$ be a set and $\varphi : E \rightarrow \mathbb{R}^m$ a Lipschitz mapping. We say that a linear mapping $L : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a *derivative of φ relative to E* at $y \in E$ if L is a unique linear mapping with the property

$$\lim_{w \rightarrow y, w \in E} \frac{\varphi(w) - \varphi(y) - L(w - y)}{|w - y|} = 0.$$

We denote this derivative by $\varphi'(y)$ even if E is not a neighborhood of y . A Lipschitz mapping $\bar{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is called an *entire Lipschitz extension* of φ if $\bar{\varphi} = \varphi$ on E .

Let us summarize some well known properties of Lipschitz mappings.

Proposition 3. *Let $E \subset \mathbb{R}^d$ be a set and $\varphi : E \rightarrow \mathbb{R}^m$ a Lipschitz mapping.*

- (a) (McShane, see [35, Prop. 2.19].) *An entire Lipschitz extension $\bar{\varphi}$ of φ always exists.*
- (b) *Let $\bar{\varphi}$ be an entire Lipschitz extension of φ and $y \in E$. If the derivatives $\varphi'(y)$ and $\bar{\varphi}'(y)$ exist, then they are equal; if y is a density point of E , then $\varphi'(y)$ exists if and only if $\bar{\varphi}'(y)$ exists.*
- (c) ([40, Section 2.10.11]) *If $F \subset E$, then $\mathcal{H}^d(\varphi(F)) \leq K^d |F|$, where K is the Lipschitz constant of φ .*
- (d) (Sard, see [35, Lemma 2.73].) *If $Z = \{y \in E : \text{rank}(\varphi'(y)) < d\}$, then $\mathcal{H}^d(\varphi(Z)) = 0$.*

- (e) (Calderón and Zygmund, see [40, Section 3.1.15], [32, Section 3.10].) *For any $\varepsilon > 0$ there exists a $\mathcal{C}^1$ mapping $\psi: \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that the measure of the set $\{y \in \mathbb{R}^d: \varphi(y) \neq \psi(y) \text{ or } \varphi'(y) \neq \psi'(y)\}$ is less than ε .*

3. LIPSCHITZ SURFACE

In this section we consider a fixed parametric Lipschitz surface $\varphi: E \rightarrow \mathbb{R}^n$. Recall that we suppose that $E \subset \mathbb{R}^{n-1}$ is Lebesgue measurable with $0 < |E| < \infty$. Let K be the Lipschitz constant of φ .

3.1. Integration over a Lipschitz surface. The image $\varphi(E)$ is an $(n-1)$ -dimensional rectifiable set. If φ is one-to-one, then the parametrization induces an *orientation* on $\varphi(E)$ (see [40, 4.1.28]), namely the unit tangent $(n-1)$ -vector field

$$\xi = \frac{\Lambda_{n-1}\varphi'}{|\Lambda_{n-1}\varphi'|}$$

defined at all points $\varphi(y)$ where the derivative $\varphi'(y)$ exists. As our co-dimension is one, we can simplify the linear algebra and identify ξ (via the Hodge star operator) with the unit normal *vector field* $\mathbf{n}_\varphi$, where

$$\mathbf{n}_\varphi(\varphi(y)) = \mathbf{n}_{\varphi,y} := \frac{N_\varphi(y)}{|N_\varphi(y)|}.$$

(Here, the terms “tangent” and “normal” are used in a weaker sense; namely, related to the approximate tangent cone, [40, Section 3.2.16].) Then by the area formula [40, Section 3.2.3], we have a representation formula for the vector valued measure $\vec{\nu}_\varphi$ induced by φ as in Definition 1, namely

$$(4) \quad \vec{\nu}_\varphi(S) = \int_{E \cap \varphi^{-1}(S)} N_\varphi(y) dy = \int_{E \cap \varphi^{-1}(S)} \mathbf{n}_{\varphi,y} |N_\varphi(y)| dy = \int_S \mathbf{n}_\varphi d\mathcal{H}^{n-1}$$

for each $\mathcal{H}^{n-1}$ -measurable $S \subset \varphi(E)$. Note that $|N_\varphi|$ is the $(n-1)$ -dimensional Jacobian of φ , [40, Section 3.2.1].

If φ fails to be one-to-one, (4) continues to hold provided that we define $\mathbf{n}_\varphi(x)$ as

$$\mathbf{n}_\varphi(x) = \sum_{y \in \varphi^{-1}(x)} \mathbf{n}_{\varphi,y}.$$

Then $\mathbf{n}_\varphi$ is no more a unit vector field in general; however, as we show below (see 3.5), for $\mathcal{H}^{n-1}$ -a.e. $x \in \varphi(E)$ we have

$$(5) \quad \mathbf{n}_\varphi(x) = \theta(x) \tilde{\mathbf{n}}_\varphi(x),$$

where $\tilde{\mathbf{n}}_\varphi(x)$ is a unit normal vector to $\varphi(E)$ at x and $\theta(x) \geq 0$ is an integer multiplicity. In general, $\theta(x)$ differs from the cardinality of $\varphi^{-1}(x)$ because of the cancellation effect.

3.2. Exceptional sets in $\mathbb{R}^{n-1}$. There are certain points of $\mathbb{R}^{n-1}$ that we want to exclude. Denote by $\partial_* E$ the essential (or “measure-theoretic”) boundary of E (it consists of the points y where the density of E at y is neither 0 nor 1, see [32], [35] or [39]) and by N the set of all points of non-differentiability of φ (in the sense “relative to E ”).

Fortunately, the sets $\partial_* E$ and N have measure zero. The set N has measure zero by the Rademacher Theorem [40, Section 3.1.6] applied to an entire extension of φ and the set $\partial_* E$ by the Lebesgue density theorem. By Proposition 3(c) the Lipschitz image $\varphi(\partial_* E \cup N)$ is of $\mathcal{H}^{n-1}$ -measure zero.

3.3. Critical values. A point $y \in E$ is a *critical point* for φ if the rank of $\varphi'(y)$ is less than $(n-1)$. We denote the set of all critical points for φ by Z . By the Sard theorem (Proposition 3(d)) applied to an entire extension of φ , the set $\varphi(Z)$ of all *critical values* for φ has $\mathcal{H}^{n-1}$ -measure zero.

In what follows, we denote $A_1 = \varphi(N \cup Z)$; it is a set of $\mathcal{H}^{n-1}$ -measure zero.

3.4. Infinite multiplicity. We say that $z \in \mathbb{R}^n$ is a *point of infinite multiplicity* for φ if $\varphi^{-1}(z)$ is an infinite set. To estimate the size of the set A_2 of the points of infinite multiplicity of φ in E , we consider separately the set A_u of points z for which the set $\varphi^{-1}(z)$ is unbounded. Since

$$A_u = \bigcap_{k=1}^{\infty} \varphi(E \setminus B(0, k)),$$

using Proposition 3(c) we can estimate

$$\mathcal{H}^{n-1}(A_u) \leq \inf_{k \in \mathbb{N}} \mathcal{H}^{n-1}(\varphi(E \setminus B(0, k))) \leq \inf_{k \in \mathbb{N}} K^{n-1} |E \setminus B(0, k)| = 0$$

as φ is Lipschitz with the constant K and E has a finite measure. If $\varphi^{-1}(z)$ is a bounded infinite set, then we can select a convergent sequence (y_j) of distinct points of E such that $\varphi(y_j) = z$ and there exists a limit $y_\infty = \lim_{j \rightarrow \infty} y_j \in \bar{E}$. If there exists $\varphi'(y_\infty)$, then necessarily y_∞ is a critical point for φ . Therefore $A_2 \setminus A_u \subset \varphi(Z)$ and using the conclusion of Subsection 3.3 we infer that $\mathcal{H}^{n-1}(A_2) = 0$.

3.5. Self-crossing of φ . We say that z is a point of self-crossing of φ if there exist $y, w \in E$ such that $\varphi(y) = \varphi(w)$, whereas $\varphi'(y)(\mathbb{R}^{n-1}) \neq \varphi'(w)(\mathbb{R}^{n-1})$. We denote the set of all points of self-crossing of φ by A_3 . By Proposition 3(e) we find C^1 mappings $\psi_k: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ such that

$$|\{y \in E: \varphi(y) \neq \psi_k(y) \text{ or } \varphi'(y) \neq \psi'_k(y)\}| < 2^{-k}, \quad k = 1, 2, \dots$$

Let us denote the set of critical points of ψ_k by Z_k . Consider $i, j \in \mathbb{N}$ and points $y \in \mathbb{R}^{n-1} \setminus Z_i$ and $w \in \mathbb{R}^{n-1} \setminus Z_j$ such that $\psi_i(y) = \psi_j(w)$ and $\psi'_i(y) \neq \psi'_j(w)$. Define $\Phi: \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ by

$$\Phi(\eta, \eta') = \psi_i(\eta) - \psi_j(\eta').$$

Obviously, $\Phi \in C^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathbb{R}^n)$, $\Phi(y, w) = 0$ and $\Phi'(y, w)$ is an $n \times (2n-2)$ matrix whose columns are

$$D_1\varphi(y), \dots, D_{n-1}\varphi(y), -D_1\varphi(w), \dots, -D_{n-1}\varphi(w).$$

As $\varphi'(y)$ and $\varphi'(w)$ have rank $(n-1)$ and their images of $\mathbb{R}^{n-1}$ are different, necessarily there exist at least n linear independent columns in $\Phi'(y, w)$; in fact, the rank of $\Phi'(y, w)$ is exactly n as the matrix has n rows. Now, we can apply the standard Implicit Function Theorem to find a neighborhood V of (y, w) in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ such that the set

$$\{(\eta, \zeta) \in V: \psi_i(\eta) = \psi_j(\zeta)\}$$

is a graph of a function of $(n-2)$ variables. This means that

$$C_{ij} := \{(\eta, \zeta) \in (\mathbb{R}^{n-1} \setminus Z_i) \times (\mathbb{R}^{n-1} \setminus Z_j): \\ \psi_i(\eta) = \psi_j(\zeta), \psi'_i(\eta) \neq \psi'_j(\zeta)\}$$

has Hausdorff dimension $(n-2)$, thus $\mathcal{H}^{n-1}(C_{ij}) = 0$ and its projection

$$C_{ij}^1 = \{\eta \in \mathbb{R}^{n-1}: C_{ij} \cap (\{\eta\} \times \mathbb{R}^{n-1}) \neq \emptyset\}$$

has $(n-1)$ -measure zero as well. Now, let us turn our attention back to φ . There is a Lebesgue null set $N_1 \subset \mathbb{R}^{n-1}$ such that for all $y \in E \setminus N_1$ we find $k \in \mathbb{N}$ such

that $\psi_k(y) = \varphi(y)$ and $\psi'_k(y) = \varphi'(y)$. Let $z \in A_3 \setminus (\varphi(N \cup N_1 \cup Z))$. Then there exist $i, j \in \mathbb{N}$ and $y \in \mathbb{R}^{n-1} \setminus Z_i$, $w \in \mathbb{R}^{n-1} \setminus Z_j$ such that

$$\psi_i(y) = \varphi(y) = z = \varphi(w) = \psi_j(w) \quad \text{and} \quad \psi'_i(y) = \varphi'(y) \neq \varphi'(w) = \psi'_j(w),$$

so that

$$z \in \bigcup_{ij} \varphi(C_{ij}^1).$$

Using Proposition 3(c) we conclude that $\mathcal{H}^{n-1}(A_3) = 0$.

3.6. Summary. We have observed that

$$(6) \quad \mathcal{H}^{n-1}(A_1 \cup A_2 \cup A_3) = 0.$$

Assume that $z \in \varphi(E)$ does not belong to this set. Then, in particular, the set $E \cap \varphi^{-1}(z)$ is finite, the derivative $\varphi'(y)$ is well defined at all points $y \in E \cap \varphi^{-1}(z)$ and there exists an $(n-1)$ -dimensional linear subspace $T_z(\varphi)$ of $\mathbb{R}^n$ such that $\varphi'(y)(\mathbb{R}^{n-1}) = T_z(\varphi)$ for each $y \in E \cap \varphi^{-1}(z)$. The unit normal vector $\mathbf{n}_\varphi(z)$ defined in 3.1 makes sense at z and admits the representation (5).

Theorem 3. *Let $u: \Omega \rightarrow \mathbb{R}$ be a precisely represented BV function and $\varphi: E \rightarrow \Omega$ an $(n-1)$ -dimensional parametric surface. Suppose that $-\vec{\nu}_\varphi = Du$. Then for $\mathcal{H}^{n-1}$ -a.e. $x \in \varphi(E)$ we have $\mathbf{n}_u(x) = -\mathbf{n}_\varphi(x)$.*

Proof. By [35, Theorem 3.77], for each Borel set $S \subset \varphi(E)$ we have

$$\int_S \mathbf{n}_u d\mathcal{H}^{n-1} = Du(S).$$

By our assumptions, $Du = -\vec{\nu}_\varphi$, so that using (4) we can continue

$$\int_S \mathbf{n}_u d\mathcal{H}^{n-1} = -\nu_\varphi(S) = -\int_S \mathbf{n}_\varphi d\mathcal{H}^{n-1}.$$

It follows that $\mathbf{n}_u = -\mathbf{n}_\varphi$ holds $\mathcal{H}^{n-1}$ -a.e. in $\varphi(E)$. $\square$

4. TYPICAL BEHAVIOR OF CURVES

4.1. Curves meeting a set. Given $A \subset \mathbb{R}^n$ and $m \in \mathbb{N} \cup \{\infty\}$, we denote by $\Gamma_m(A)$ the family of all paths $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ which meet A at least m -times, this means,

$$\#\{t \in [0, \ell]: \gamma(t) \in A\} \geq m$$

where $\#(A)$ stands for the cardinality of a set A . We write $\Gamma(A) = \Gamma_1(A)$.

Theorem 4. *Let $A \subset \mathbb{R}^n$. Then*

$$AM(\Gamma_m(A)) \leq \frac{C}{m} \mathcal{H}^{n-1}(A)$$

where C depends only on the dimension n .

Proof. For $m = 1$ see [52, Theorem 3.17]. Let (ρ_j) be an admissible sequence for $\Gamma_1(A)$. We can find a partition

$$0 = t_0 < t_1 < \dots < t_m = \ell$$

such that γ meets A on each $[t_{i-1}, t_i]$, so that the part (reparametrized as from $[0, t_i - t_{i-1}]$) belongs to $\Gamma_1(A)$. Given $0 < \varepsilon < 1$, we can find an integer p_i such that

$$\int_{t_{i-1}}^{t_i} \rho_j(\gamma(t)) dt > 1 - \varepsilon, \quad j \geq p_i,$$

so that

$$\int_0^\ell \frac{\rho_j(\gamma(t))}{m} dt > 1 - \varepsilon, \quad j \geq \max\{p_1, \dots, p_m\}.$$

Therefore the sequence (ρ_j/m) is admissible for $\Gamma_m(A)$ and the assertion follows. $\square$

Corollary 1. *Let $A \subset \mathbb{R}^n$, $\mathcal{H}^{n-1}(A) = 0$. Then*

$$AM(\Gamma(A)) = 0.$$

Corollary 2. *Let $A \subset \mathbb{R}^n$, $\mathcal{H}^{n-1}(A) < \infty$. Then*

$$AM(\Gamma_\infty(A)) = 0.$$

Corollary 3. *Let $\varphi: E \rightarrow \mathbb{R}^n$ be a Lipschitz parametric surface. Then for AM-a.e. path $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$, the set*

$$\{(y, t) \in E \times [0, \ell]: \gamma(t) = \varphi(y)\}$$

is finite.

Proof. Of course, $\mathcal{H}^{n-1}(\varphi(E)) < \infty$. Let A_2 be as in Section 3. If γ does not meet A_2 and does not belong to $\Gamma_\infty(\varphi(E))$, then there is a finite set $T \subset [0, \ell]$ such that the set

$$\{y \in E: \gamma(t) = \varphi(y)\}$$

is finite for $t \in T$ and empty for $t \notin T$. Therefore (6), Corollary 1 and Corollary 2 give the assertion. $\square$

4.2. Tangential curves.

Definition 2. Let $A \subset \mathbb{R}^n$ and $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ be a path (parametrized by arclength). We say that γ is *right tangential* to A at $t \in [0, \ell]$ if $\gamma(t) \in A$ and

$$\lim_{s \rightarrow t+} \frac{\text{dist}(\gamma(s), A)}{s - t} = 0.$$

We say that γ is *left tangential* to A at $t \in (0, \ell]$ if $\gamma(t) \in A$ and

$$\lim_{s \rightarrow t-} \frac{\text{dist}(\gamma(s), A)}{s - t} = 0.$$

We say that γ is *tangential* to A if there exists $t \in (0, \ell]$ such that γ is left or right tangential to A at t . We denote the family of all curves tangential to A by $\Gamma_\tau(A)$.

Definition 3. A set $S \subset \mathbb{R}^n$ is called a $\mathcal{C}^1$ *surface* (“non-parametric”) if there exist an open set $G \subset \mathbb{R}^{n-1}$ and a $\mathcal{C}^1$ parametric surface $\psi: G \rightarrow \mathbb{R}^n$ such that the rank of ψ' is $(n-1)$ in G , $S = \psi(G)$ and $\psi: G \rightarrow S$ is a homeomorphism.

Remark 2. In spite of language similarity between the concepts of surface and parametric surface, the meanings are completely different. Surface is a set, whereas parametric surface is a mapping, and in addition, it is required neither to be one-to-one, nor to be “regular” (this means, to satisfy the $(n-1)$ -rank condition of φ').

Proposition 4. [42, Theorem 47] *Let $S \subset \mathbb{R}^n$ be a $\mathcal{C}^1$ -surface. Then*

$$AM(\Gamma_\tau(S)) = 0.$$

4.3. Constancy theorem.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u: \Omega \rightarrow \mathbb{R}$ be a precisely represented BV function and $\varphi: E \rightarrow \Omega$ be an $(n-1)$ -dimensional parametric surface. Suppose that $\vec{\nu}_\varphi = -Du$. Then for AM-a.e. path $\gamma: [0, \ell] \rightarrow \Omega$ it holds that either $\gamma \in \Gamma(\varphi(E))$ or $u \circ \gamma$ is constant.*

Proof. For $\varepsilon > 0$, we define Γ_ε as the set of all paths $\gamma: [0, \ell] \rightarrow \Omega$ such that $\gamma \notin \Gamma(\varphi(E))$ and $|u(\gamma(\ell)) - u(\gamma(0))| \geq \varepsilon$. We claim that $AM(\Gamma_{1/m}) = 0$ for each $m \in \mathbb{N}$. To prove the claim, we choose $\alpha \in (0, 1/m)$, we find a compact set $H \subset \varphi(E)$ such that $|Du|(\Omega \setminus H) < \alpha^2$, and we set

$$\mu = |Du|_{|\Omega \setminus H}.$$

Let $(\omega_\delta)_{\delta>0}$ be a standard family of mollifiers, i.e. for $x \in \mathbb{R}^n$

$$\omega_\delta \in \mathcal{C}_c^\infty(\mathbb{R}^n)^+, \text{ spt } \omega_\delta \subset \overline{B}(0, \delta), \int_{\mathbb{R}^n} \omega_\delta(y) dy = 1 \text{ and } \omega_\delta(x) = \delta^{-n} \omega_1\left(\frac{x}{\delta}\right).$$

Set

$$\rho_k = m \omega_{1/k} * \mu, \quad k = 1, 2, \dots$$

Then

$$(7) \quad \|\rho_k\|_{L^1(\mathbb{R}^n)} \leq m \mu(\mathbb{R}^n) < m \alpha^2 < \alpha.$$

We recall that $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ (see Subsection 2.2), and thus by Corollary 1,

$$(8) \quad AM(\Gamma_{1/m}) = AM(\tilde{\Gamma}_{1/m}), \quad \text{where } \tilde{\Gamma}_{1/m} = \Gamma_{1/m} \setminus \Gamma(S_u \setminus J_u).$$

Our next step is to show that the sequence (ρ_k) is admissible for $\tilde{\Gamma}_{1/m}$. Choose $\gamma \in \tilde{\Gamma}_{1/m}$. By [35, Theorem 3.80], the mollified functions $\omega_\delta * u$ converge pointwise to u on $\Omega' \setminus (S_u \setminus J_u)$ for each $\Omega' \subset\subset \Omega$. In particular,

$$(\omega_{1/k} * u)(\gamma(0)) \rightarrow u(\gamma(0)) \quad \text{and} \quad (\omega_{1/k} * u)(\gamma(\ell)) \rightarrow u(\gamma(\ell)).$$

If k is such that

$$H \cap \bigcup_{x \in \langle \gamma \rangle} B(x, 1/k) = \emptyset,$$

then $\omega_{1/k} * |Du| = \omega_{1/k} * \mu$ on $\langle \gamma \rangle$. Hence

$$\begin{aligned} \frac{1}{m} &\leq |u(\gamma(\ell)) - u(\gamma(0))| \leq \liminf_{k \rightarrow \infty} |(\omega_{1/k} * u)(\gamma(\ell)) - (\omega_{1/k} * u)(\gamma(0))| \\ &\leq \liminf_{k \rightarrow \infty} \int_{\gamma} |D(\omega_{1/k} * u)| ds \leq \liminf_{k \rightarrow \infty} \int_{\gamma} \omega_{1/k} * |Du| ds \\ &= \liminf_{k \rightarrow \infty} \int_{\gamma} \omega_{1/k} * \mu ds = \frac{1}{m} \liminf_{k \rightarrow \infty} \int_{\gamma} \rho_k ds. \end{aligned}$$

This shows that (ρ_k) is admissible for $\tilde{\Gamma}_{1/m}$ and thus, by (7) and (8),

$$AM(\Gamma_{1/m}) = AM(\tilde{\Gamma}_{1/m}) \leq \alpha.$$

Letting $\alpha \rightarrow 0$ we obtain $AM(\Gamma_{1/m}) = 0$ and using subadditivity of the approximation modulus we conclude that $u(\gamma(\ell)) - u(\gamma(0)) = 0$ for AM -a.e. path $\gamma \notin \Gamma(\varphi(E))$. $\square$

4.4. Entries and exits.

Definition 4. Let $\varphi: E \rightarrow \mathbb{R}^n$ be an $(n-1)$ -dimensional parametric surface and $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ be a path (parametrized by arclength). Let $(y, t) \in E \times [0, \ell]$ and assume that $\gamma(t) = \varphi(y) = x$ and $\mathbf{n} := \mathbf{n}_{\varphi, y}$ is well defined. We say that:

- (y, t) is a couple of *outer exit* if $t < \ell$ and

$$\liminf_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \geq 0 \text{ and } \limsup_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} > 0.$$

The set of all couples of outer exit is denoted by $C^+(\varphi, \gamma)$.

- (y, t) is a couple of *inner exit* if $t < \ell$ and

$$\liminf_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} < 0 \text{ and } \limsup_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \leq 0.$$

The set of all couples of inner exit is denoted by $C_+(\varphi, \gamma)$.

- (y, t) is a couple of *outer entry* if $t > 0$ and

$$\liminf_{s \rightarrow t_-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} < 0 \text{ and } \limsup_{s \rightarrow t_-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \leq 0.$$

The set of all couples of outer entry is denoted by $C^-(\varphi, \gamma)$.

- (y, t) is a couple of *inner entry* if $t > 0$ and

$$\liminf_{s \rightarrow t_-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \geq 0 \text{ and } \limsup_{s \rightarrow t_-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} > 0.$$

The set of all couples of inner entry is denoted by $C_-(\varphi, \gamma)$.

Remark 3. Let φ, γ, y, t be as in the definition above. In addition, suppose that $0 < t < \ell$ and γ is differentiable at t . Then (y, t) is a couple of outer exit if and only if (y, t) is a couple of inner entry, and this is if and only if

$$\det(\gamma'(t), D_1\varphi(y), \dots, D_{n-1}\varphi(y)) > 0.$$

Similarly, (y, t) is a couple of inner exit if and only if (y, t) is a couple of outer entry, and this is if and only if

$$\det(\gamma'(t), D_1\varphi(y), \dots, D_{n-1}\varphi(y)) < 0.$$

Remark 4. The terminology of Definition 4 is motivated by a model example in which φ parametrizes the essential boundary of a BV set U . Then the normal $\mathbf{n}$ is outward with respect to U and we distinguish whether we exit inward or outward.

Definition 5 (Regular crossing; crossing number). Let the sets A_1, A_2, A_3 have the same meaning as in Section 3. We say that γ has *regular crossing* with φ if the following properties hold:

- (rc1) The number of couples $(y, t) \in E \times [0, \ell]$ such that $\gamma(t) = \varphi(y)$ is finite.
- (rc2) $\langle \gamma \rangle \cap \varphi(E) \cap (A_1 \cup A_2 \cup A_3) = \emptyset$.
- (rc3) If $(y, t) \in E \times [0, \ell]$ and $\gamma(t) = \varphi(y)$, then (y, t) is either a couple of inner exit or a couple of outer exit.
- (rc4) If $(y, t) \in E \times (0, \ell]$ and $\gamma(t) = \varphi(y)$, then (y, t) is either a couple of inner entry or a couple of outer entry.

The family of all curves γ which have regular crossing with φ is denoted by Γ_φ . If $\gamma \in \Gamma_\varphi$, we define the *crossing number* of γ with φ as

$$(9) \quad \otimes(\varphi, \gamma) = \frac{1}{2}(\#C^+(\varphi, \gamma) + \#C_-(\varphi, \gamma) - \#C^-(\varphi, \gamma) - \#C_+(\varphi, \gamma)).$$

Lemma 1. Let $\varphi: E \rightarrow \mathbb{R}^n$ be an $(n-1)$ -dimensional parametric surface. Then for AM-a.e. path $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ the following holds: If $(y, t) \in E \times [0, \ell]$ and $\gamma(t) = \varphi(y)$, then

$$(10) \quad \limsup_{s \rightarrow t_+} \frac{|(\gamma(s) - \gamma(t)) \cdot \mathbf{n}_{\varphi, y}|}{s - t} > 0.$$

Proof. Let $\psi_j: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ be C^1 mappings such that

$$|\{y \in E: \varphi(y) \neq \psi_j(y) \text{ or } \varphi'(y) \neq \psi_j'(y)\}| < 2^{-j},$$

see Proposition 3 (e). Then there is a Lebesgue null set $N_1 \subset \mathbb{R}^{n-1}$ such that for all $y \in E \setminus N_1$ we find $j \in \mathbb{N}$ such that $\psi_j(y) = \varphi(y)$ and $\psi_j'(y) = \varphi'(y)$. Denote

$A_4 = \varphi(N_1)$; this is a $\mathcal{H}^{n-1}$ -null set. Let A_1, A_2, A_3 be as in Section 3. For each $j \in \mathbb{N}$, let $(B_m^j)_m$ be a sequence of all balls with rational centers and rational radii in $\mathbb{R}^{n-1}$ with the property that $S_m^j := \psi_j(B_m^j)$ is a $\mathcal{C}^1$ surface. Consider a path $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ such that $\gamma \notin \Gamma(A_1 \cup A_2 \cup A_3 \cup A_4)$, i.e. γ does not meet the set $A_1 \cup A_2 \cup A_3 \cup A_4$, and a couple $(y, t) \in E \times [0, \ell)$ such that $\gamma(t) = \varphi(y)$. Then there exists $j \in \mathbb{N}$ such that $\psi_j(y) = \varphi(y)$ and $\psi_j'(y) = \varphi'(y)$. Since y is not a critical point for $\bar{\varphi}$, we can find $m \in \mathbb{N}$ such that $y \in B_m^j$. Assume that $\gamma \notin \Gamma_\tau(S_m^j)$. Let $s \in (t, \ell)$. Let x be the orthogonal projection of $\gamma(s) - \gamma(t)$ to $\psi_j'(y)(\mathbb{R}^{n-1})$. Find $w \in \mathbb{R}^{n-1}$ such that $\psi_j'(y)(w - y) = x$. If s is close enough to t , then $w \in B_m^j$. We estimate

$$\begin{aligned}
 (11) \quad & \frac{\text{dist}(\gamma(s), S_m^j)}{s-t} \leq \frac{|\gamma(s) - \psi_j(w)|}{s-t} \\
 & \leq \frac{|\gamma(s) - \gamma(t) - x|}{s-t} + \frac{|\psi_j(w) - \psi_j(y) - \psi_j'(y)(w - y)|}{|w - y|} \frac{|w - y|}{s-t} \\
 & = \frac{|(\gamma(s) - \gamma(t)) \cdot \mathbf{n}_{\varphi, y}|}{s-t} + \frac{|\psi_j(w) - \psi_j(y) - \psi_j'(y)(w - y)|}{|w - y|} \frac{|w - y|}{s-t}.
 \end{aligned}$$

Note that $\varphi'(y)$ is an isomorphism of $\mathbb{R}^{n-1}$ onto $\varphi'(y)(\mathbb{R}^{n-1})$, and thus there exists a constant $\lambda > 0$ such that

$$\lambda|w - y| \leq |\varphi'(y)(w - y)| = |x| \leq |\gamma(s) - \gamma(t)| \leq s - t.$$

Taking into account the definition of derivative, it follows that the second term in (11) tends to zero as $s \rightarrow t+$. Since by (6), Corollary 1 and Proposition 4

$$AM(\Gamma(A_1 \cup A_2 \cup A_3 \cup A_4)) = 0, \quad AM(\Gamma_\tau(\psi_j(B_m^j))) = 0, \quad j, m \in \mathbb{N},$$

and since the AM -modulus is countably subadditive, the assertion holds. $\square$

Theorem 6. *Let $\varphi: E \rightarrow \mathbb{R}^n$ be an $(n-1)$ -dimensional parametric surface. Then AM -a.e. path has a regular crossing with φ .*

Proof. We can exclude the paths violating Properties (rc1) and (rc2) by (6), Corollary 1 and Corollary 3. The rest of the proof is devoted to Property (rc3). By a similarity argument, this verifies also (rc4).

Let $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ satisfy the conclusion of Lemma 1. Let $\bar{\varphi}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ be an entire Lipschitz extension of φ and $G \subset \mathbb{R}^{n-1}$ be an open set containing E such that $|G| < \infty$. Consider a point $(y, t) \in E \times [0, \ell)$ such that $\gamma(t) = \varphi(y) =: z$. Assume that $z \notin A_1 \cup A_2 \cup A_3$. Denote $\mathbf{n} = \mathbf{n}_{\varphi, y}$. Let ψ be the tangent mapping

$$\psi(w) = \varphi(y) + \varphi'(y)(w - y)$$

and b be the bi-Lipschitz constant of ψ . Then by (10) there exist $\varepsilon \in (0, \frac{1}{2b})$ and $s_i > t$ such that $s_i \searrow t$ and

$$\frac{|(\gamma(s_i) - \gamma(t)) \cdot \mathbf{n}|}{|\gamma(s_i) - \gamma(t)|} \geq \frac{|(\gamma(s_i) - \gamma(t)) \cdot \mathbf{n}|}{s_i - t} > 2\varepsilon.$$

On the other hand, using differentiability and regularity of $\bar{\varphi}$ at y we easily observe that there exists $\delta > 0$ such that $B(y, \delta) \subset G$ and

$$(12) \quad 0 < |w - y| < \delta \implies \begin{cases} \frac{|\bar{\varphi}(w) - \psi(w)|}{|w - y|} = \frac{|\bar{\varphi}(w) - \bar{\varphi}(y) - \bar{\varphi}'(y)(w - y)|}{|w - y|} < \varepsilon, \\ \frac{|(\bar{\varphi}(w) - \bar{\varphi}(y)) \cdot \mathbf{n}|}{|\bar{\varphi}(w) - \bar{\varphi}(y)|} < \varepsilon. \end{cases}$$

Fix $r \in (0, \delta/b)$. Let

$$U = \{(w, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\psi(w) - z|^2 + \tau^2 < r^2\},$$

$$U^+ = \{(w, \tau) \in U : \tau > 0\}.$$

and note that $|w - y| < br < \delta$ whenever $(w, \tau) \in U$. Define the homotopy mapping $h : \overline{U} \times [0, 1] \rightarrow \mathbb{R}^n$ by

$$h(w, \tau, \lambda) = \psi(w) + \lambda(\bar{\varphi}(w) - \psi(w)) + \tau \mathbf{n},$$

see Subsection 2.3. Using (12) we estimate

$$(13) \quad \begin{aligned} |h(w, \tau, \lambda) - z| &\geq |h(w, \tau, 0) - z| - |h(w, \tau, \lambda) - h(w, \tau, 0)| \\ &> r - \lambda|\psi(w) - \bar{\varphi}(w)| \\ &\geq r - \varepsilon|w - y| \geq r/2, \quad (w, \tau) \in \partial U, \end{aligned}$$

and

$$(14) \quad \frac{(h(w, 0, \lambda) - z) \cdot \mathbf{n}}{|h(w, 0, \lambda) - z|} \leq \frac{(\bar{\varphi}(w) - z) \cdot \mathbf{n}}{|\bar{\varphi}(w) - z|} < \varepsilon, \quad (w, 0) \in \partial U^+.$$

Consider $s_i > t$ such that

$$(15) \quad \frac{(\gamma(s_i) - z) \cdot \mathbf{n}}{|\gamma(s_i) - z|} > 2\varepsilon \quad \text{and} \quad |\gamma(s) - z| < r/2, \quad t < s < s_i.$$

Then $\gamma(s_i) \in h(U, 0)$ and by (13) and (14) we have

$$(16) \quad (w, \tau, \lambda) \in \partial U^+ \times [0, 1] \implies \gamma(s_i) \neq h(w, \tau, \lambda).$$

Hence a homotopy argument, see Subsection 2.3, shows that

$$\deg(h(\cdot, 1), U^+, \gamma(s_i)) = \deg(h(\cdot, 0), U^+, \gamma(s_i)) = 1.$$

Similarly we can prove that if $s_j > t$,

$$(17) \quad \frac{(\gamma(s_j) - z) \cdot \mathbf{n}}{|\gamma(s_j) - z|} < -2\varepsilon \quad \text{and} \quad |\gamma(s) - z| < r/2, \quad t < s < s_j,$$

and then it follows that

$$\deg(h(\cdot, 1), U^+, \gamma(s_j)) = 0.$$

We infer that if both (15) and (17) occur, then there exist s between s_i and s_j and $(w, \tau) \in \partial U^+$ such that $\gamma(s) = h(w, \tau, 1)$. By (13), $\tau = 0$ and thus $\gamma(s) = \bar{\varphi}(w)$.

We have shown that the case

$$(18) \quad \liminf_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} < 0 < \limsup_{s \rightarrow t_+} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t}$$

implies infinite number of intersections of γ with $\bar{\varphi}(G)$. However, by Corollary 2, $AM(\Gamma_\infty(\bar{\varphi}(G))) = 0$. Taking into account the result of Lemma 1, we conclude that AM -a.e. γ satisfies (rc3). $\square$

5. ANALYSIS OF EXISTS AND ENTRIES

In the following lemma, the projections Π_i are defined as in Proposition 1.

Lemma 2. *Let $X, Y \subset Q := [0, 1]^n$. Suppose that $|X|, |Y| \geq \frac{1}{2}$ and $X \cup Y = Q$. Then there exists $i \in \{1, \dots, n\}$ such that*

$$\mathcal{H}^{n-1}(\Pi_i(X) \cap \Pi_i(Y)) > \frac{1}{4n}.$$

Proof. For $i \in \{1, \dots, n\}$, we define

$$Z_i = \{(x, y) \in Q^2 : (x_1, \dots, x_i, y_{i+1}, \dots, y_n) \in X, \\ (x_1, \dots, x_{i-1}, y_i, \dots, y_n) \in Y\}.$$

We can imagine that we travel from $y \in Y$ to $x \in X$ through $z_0, z_1, \dots, z_m$, where

$$z_i := (x_1, \dots, x_i, y_{i+1}, \dots, y_n),$$

so that $z_0 = y$ and $z_n = x$. Since each z_i belongs to X or Y , there exists $i \in \{1, \dots, n\}$ such that $z_{i-1} \in Y$ and $z_i \in X$, which means exactly that $(x, y) \in Z_i$. As we can join every $y \in Y$ with every $x \in X$ this way, it follows that $X \times Y \subset \bigcup_i Z_i$, so that there exists i such that

$$\mathcal{L}^{2n}(Z_i) \geq \frac{1}{n} \mathcal{L}^{2n}(X \times Y) \geq \frac{1}{4n}.$$

Now, define the projection

$$\Sigma_i(x, y) = (x_1, \dots, x_{i-1}, 0, y_{i+1}, \dots, y_n), \quad (x, y) \in Q \times Q.$$

Then

$$\Sigma_i(Z_i) \subset \Pi_i(X) \cap \Pi_i(Y)$$

and thus

$$\mathcal{H}^{n-1}(\Pi_i(X) \cap \Pi_i(Y)) \geq \mathcal{H}^{n-1}(\Sigma_i(Z_i)) \geq \mathcal{L}^{2n}(Z_i) \geq \frac{1}{4n}.$$

□

Definition 6. Let $u: \Omega \rightarrow \mathbb{R}$ be a precisely represented BV function and $\varphi: E \rightarrow \Omega$ be an $(n-1)$ -dimensional parametric surface. Suppose that $\vec{\nu}_\varphi = -Du$. We say that a path $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ has *property (C)* if

- $\gamma \in \Gamma_\varphi$ (see Definition 5),
- $\mathbf{n}_\varphi(x) = -\mathbf{n}_u(x)$ at all $x \in \langle \gamma \rangle \cap \varphi(E)$,
- If $I \subset [0, \ell]$ is an interval such that $\gamma(I) \cap \varphi(E) = \emptyset$, then $u \circ \gamma$ is constant on I ,
- if (y, t) is a couple of outer exit, then there exists $\delta > 0$ such that $u \circ \gamma(s) = b$ on $(t, t + \delta)$, where b is the L^1 -approximate limit of u along the halfspace

$$\mathbb{H}^+ := \{x \in \mathbb{R}^n : (x - z) \cdot \mathbf{n}_{\varphi, y} > 0\},$$

- if (y, t) is a couple of inner exit, then there exists $\delta > 0$ such that $u \circ \gamma(s) = a$ on $(t, t + \delta)$, where a is the L^1 -approximate limit of u along the halfspace

$$\mathbb{H}^- := \{x \in \mathbb{R}^n : (x - z) \cdot \mathbf{n}_{\varphi, y} < 0\},$$

- if (y, t) is a couple of outer entry, then there exists $\delta > 0$ such that $u \circ \gamma(s) = b$ on $(t - \delta, t)$, where b is the L^1 -approximate limit of u along the halfspace $\mathbb{H}^+$,
- if (y, t) is a couple of inner entry, then there exists $\delta > 0$ such that $u \circ \gamma(s) = a$ on $(t - \delta, t)$, where a is the L^1 -approximate limit of u along the halfspace $\mathbb{H}^-$.

Lemma 3. Let $u: \Omega \rightarrow \mathbb{R}$ be a precisely represented BV function and $\varphi: E \rightarrow \Omega$ be an $(n-1)$ -dimensional parametric surface. Suppose that $\vec{\nu}_\varphi = -Du$. Then *AM-a.e.* path $\gamma \in \Gamma_\varphi$ has *property (C)*.

Proof. By Theorems 3, 5 and 6 and in view of Corollary 1 and (3.6) it remains to prove last four items, but because of their similarity it is enough to prove one of them. So, we will concentrate on the case of outer exit. Let $\tilde{\varphi}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ be an entire Lipschitz extension of φ and $G \subset \mathbb{R}^{n-1}$ be an open set such that $E \subset G$ and

$|G| < \infty$. Let K be the Lipschitz constant of $\bar{\varphi}$. Consider the finite Borel measure μ defined as the push-forward of the Lebesgue measure on G via $\bar{\varphi}$, this means

$$\mu(M) = |G \cap \bar{\varphi}^{-1}(M)|$$

whenever M is a Borel subset of $\mathbb{R}^n$. We define

$$R_\tau(x, \mu) = \inf\{r > 0: \mu(B(x, r)) \geq \tau r^{n-1}\}, \quad x \in \mathbb{R}^n,$$

where $\tau > 0$ is to be specified later. We want to use the result [42, Theorem 10], which states (for a given finite Borel measure μ) that the family of paths $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ for which there exists $t \in [0, \ell]$ with

$$(19) \quad \lim_{s \rightarrow t+} \frac{R_\tau(\gamma(s), \mu)}{s - t} = 0$$

has zero AM -modulus. Roughly speaking, condition (19) means that the path γ is tangential to $\bar{\varphi}(G)$ in the measure theoretic sense.

Let A be the set of all points of $x \in \Omega$ which either belong to $S_u \setminus J_u$ or belong to $\varphi(E)$ and violate $\mathbf{n}_u(x) = -\mathbf{n}_\varphi(x)$. Then by Subsection 2.2 and Theorem 3, $\mathcal{H}^{n-1}(A) = 0$. Let $\gamma \notin \Gamma(A)$ and assume that γ meets the constancy behavior of Theorem 5. Let (y, t) be a couple of outer exit and $z := \varphi(y) = \gamma(t)$. Since $z \notin A$, either z is an L^1 -Lebesgue point for u or it is a jump point and $\mathbf{n}_u(z) = -\mathbf{n}_\varphi(z)$. Anyway, there exists an approximate limit α of u along the halfspace $\mathbb{H}^+$. We find $t' > t$ such that $\gamma((t, t')) \cap \varphi(E) = \emptyset$. Then $u \circ \gamma$ is constant on (t, t') . Assume that this constant is some $\beta > \alpha$. Choose $c \in (\alpha, \beta)$ and set

$$M_1 = \{x \in \mathbb{H}^+ : u(x) < c\}, \quad M_2 = \{x \in \mathbb{H}^+ : u(x) \geq c\}.$$

Note that the set M_2 has Lebesgue density zero at z . Choose $\varepsilon > 0$,

$$(20) \quad \varepsilon < \min\{1, 1/K\}$$

There exists $\delta_1 > 0$ such that for all $r \in (0, \delta_1)$ we have

$$(21) \quad \frac{|M_2 \cap Q(\gamma(t), r)|}{|Q(\gamma(t), r)|} < \frac{\varepsilon^n}{2^{n+1}}.$$

We know that $\bar{\varphi}$ is differentiable at y ; let ψ be the tangent mapping

$$\psi(w) = z + \varphi'(y)(w - y), \quad w \in \mathbb{R}^{n-1}.$$

There exists $\delta_2 > 0$ such that

$$(22) \quad |\psi(w) - z| < \delta_2 \implies (w \in G \text{ and } |\bar{\varphi}(w) - \psi(w)| \leq \varepsilon |\psi(w) - z|).$$

Since (y, t) is a point of outer exit, there exists $\delta_3 > 0$ such that

$$(23) \quad t < s < t + \delta_3 \implies (\gamma(s) - z) \cdot \mathbf{n}_{\varphi, y} > -\varepsilon \sqrt{n} (s - t)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and fix $s > t$ such that $s < \min\{t + \delta/2, t'\}$. We distinguish two cases.

CASE 1. Let

$$(24) \quad |(\gamma(s) - z) \cdot \mathbf{n}_{\varphi, y}| < \varepsilon \sqrt{n} (s - t).$$

Let x be the orthogonal projection of $\gamma(s)$ on $\psi(\mathbb{R}^{n-1})$. Denote $v = \psi^{-1}(x)$, $\kappa = (K + \sqrt{n} + 2)^{-1}$ and $r = \varepsilon(s - t)/\kappa$. If $w \in B(v, \kappa r) = B(v, \varepsilon(s - t))$, then by (20)

$$(25) \quad \begin{aligned} |\psi(w) - x| &\leq K|w - v| \leq K\varepsilon(s - t) < (s - t), \\ |\psi(w) - z| &\leq |\psi(w) - x| + |x - z| \leq (s - t) + |\gamma(s) - z| \leq 2(s - t) < \delta. \end{aligned}$$

From (22), (24) and (25) we obtain that $w \in G$ and

$$\begin{aligned} |\varphi(w) - \gamma(s)| &\leq |\varphi(w) - \psi(w)| + |\psi(w) - x| + |x - \gamma(s)| \\ &\leq \varepsilon |\psi(w) - z| + K\varepsilon(s-t) + |(\gamma(s) - z) \cdot \mathbf{n}_{\varphi, y}| \\ &< \varepsilon(2 + K + \sqrt{n})(s-t) = r. \end{aligned}$$

Hence there is τ_1 depending only on n and K such that

$$(26) \quad \mu(B(\gamma(s), r)) \geq |B(v, \kappa r)| \geq \tau_1 r^{n-1}.$$

For any $\tau \in (0, \tau_1]$ we have

$$(27) \quad R_\tau(\gamma(s)) \leq r = \varepsilon(s-t)/\kappa.$$

CASE 2. Let

$$(28) \quad (\gamma(s) - z) \cdot \mathbf{n}_{\varphi, y} \geq \varepsilon \sqrt{n}(s-t).$$

Then $Q(\gamma(s), \varepsilon(s-t)) \subset Q(\gamma(t), 2(s-t))$ and thus

$$\frac{|M_2 \cap Q(\gamma(s), \varepsilon(s-t))|}{|Q(\gamma(s), \varepsilon(s-t))|} \leq \frac{|M_2 \cap Q(\gamma(t), 2(s-t))|}{\varepsilon^n 2^{-n} |Q(\gamma(t), 2(s-t))|} \leq \frac{1}{2}.$$

On the other hand, $\gamma(s)$ belongs to the essential interior of M_2 , as all points $\gamma(s)$ are Lebesgue points of u . Thus

$$\lim_{\tau \rightarrow 0+} \frac{|M_2 \cap Q(\gamma(s), \tau)|}{|Q(\gamma(s), \tau)|} = 1.$$

Therefore there exists $0 < \sigma \leq \varepsilon(s-t)$ such that

$$\frac{|M_2 \cap Q(\gamma(s), \sigma)|}{|Q(\gamma(s), \sigma)|} = \frac{1}{2} \quad \text{and} \quad \frac{|M_1 \cap Q(\gamma(s), \sigma)|}{|Q(\gamma(s), \sigma)|} = \frac{1}{2}.$$

Denote $Q := Q(\gamma(s), \sigma)$. By (28), $Q \subset \mathbb{H}^+$ and thus $u < c$ on $M_1 \cap Q$ and $u \geq c$ on $M_2 \cap Q$. By Lemma 2, there exists $i \in \{1, \dots, n\}$ such that

$$\mathcal{H}^{n-1}(\Pi_i(M_1 \cap Q) \cap \Pi_i(M_2 \cap Q)) > \frac{1}{4n}(2\sigma)^{n-1},$$

where Π_i denotes the orthogonal projection onto $\{x \in \mathbb{R}^n : x_i = 0\}$. As u is constant along AM -a.e. curve which does not meet $\varphi(E)$, see Theorem 5, by Proposition 1 it is constant along a.e. segment in Q orthogonal to the i -th coordinate axis which does not meet the surface $\varphi(E)$. For each $\hat{x} \in \Pi_i(M_1 \cap Q) \cap \Pi_i(M_2 \cap Q)$ this means that (as the constancy is violated) the segment $\{x \in Q : \Pi_i(x) = \hat{x}\}$ meets $\varphi(E)$. We estimate

$$\begin{aligned} K^{n-1}\mu(Q) &\geq K^{n-1}|\varphi^{-1}(Q)| \geq \mathcal{H}^{n-1}(\varphi(E) \cap Q) \geq \mathcal{H}^{n-1}(\Pi_i(\varphi(E) \cap Q)) \\ &\geq \frac{1}{4n}(2\sigma)^{n-1}. \end{aligned}$$

Hence there exists $\tau_2 > 0$ depending only on n such that for $r = \sqrt{n}\sigma$ we have

$$\mu(B(\gamma(s), r)) \geq \mu(Q) \geq \tau_2 r^{n-1}.$$

For any $\tau \in (0, \tau_1]$ we have

$$(29) \quad R_\tau(\gamma(s)) \leq r = \sqrt{n}\sigma \leq \varepsilon \sqrt{n}(s-t).$$

We have shown that the relevant estimate holds in both cases. Indeed, from (27) and (29) we obtain that (19) holds if $u \circ \gamma = \beta$ on (t, t') and $\beta > \alpha$. Similarly we can exclude paths for which $\beta < \alpha$. Hence for AM -almost all paths the required behavior holds. $\square$

6. PROOF OF THE MAIN THEOREM: EUCLIDEAN SETTING

Proof of Theorem 1. (i) $\implies$ (ii). We may assume that u is precisely represented. Let $\gamma: [0, \ell] \rightarrow \Omega$ be a path which has the property (C) of Definition 6. Then there is a partition

$$0 = t_0 < t_1 < \cdots < t_m = \ell$$

of $[0, \ell]$ such that $\gamma((t_{i-1}, t_i)) \cap \varphi(E) = \emptyset$ and $u \circ \gamma$ is constant on the intervals (t_{i-1}, t_i) . If $\gamma(t_i) \notin \varphi(E)$, then $u \circ \gamma$ is continuous at t_i and nothing happens. If $z := \gamma(t_i) \in \varphi(E)$, then there are a unit “normal” vector $\mathbf{n}$ at z and values $u^+(z), u^-(z)$ such that the approximate limit of u at z along $\mathbb{H}^+$ is $u^+(z)$ and the approximate limit of u at z along $\mathbb{H}^-$ is $u^-(z)$ where

$$\mathbb{H}^+ = \{x \in \mathbb{R}^n : (x - z) \cdot \mathbf{n} > 0\}, \mathbb{H}^- = \{x \in \mathbb{R}^n : (x - z) \cdot \mathbf{n} < 0\}.$$

Since

$$(u^+(z) - u^-(z))\mathbf{n} = \mathbf{n}_u(z) = -\mathbf{n}_\varphi(z),$$

we have

$$u^+(z) - u^-(z) = \#\{y \in \varphi^{-1}(z) : \mathbf{n}_{\varphi,y} = -\mathbf{n}\} - \#\{y \in \varphi^{-1}(z) : \mathbf{n}_{\varphi,y} = \mathbf{n}\}.$$

Concerning the behavior of γ at t_i from the left (if $t_i > 0$), we distinguish two cases. If

$$(30) \quad \liminf_{s \rightarrow t_i^-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} < 0 \quad \text{and} \quad \limsup_{s \rightarrow t_i^-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \leq 0,$$

then

$$\lim_{s \rightarrow t_i^-} u(\gamma(s)) = u^-(z)$$

and given $y \in \varphi^{-1}(z)$, then (t_i, y) is a couple of inner entry if $\mathbf{n}_{\varphi,y} = -\mathbf{n}$ and it is a couple of outer entry if $\mathbf{n}_{\varphi,y} = \mathbf{n}$. Hence, the entry at t_i contributes to the crossing number by the value

$$\frac{1}{2}(\#\{y \in \varphi^{-1}(z) : \mathbf{n}_{\varphi,y} = -\mathbf{n}\} - \#\{y \in \varphi^{-1}(z) : \mathbf{n}_{\varphi,y} = \mathbf{n}\}),$$

which is exactly $-\frac{1}{2}(u^+(z) - u^-(z))$. Then, taking into account that the precise value of u at z is the arithmetic mean of $u^+(z)$ and $u^-(z)$, we obtain that this contribution equals $-u(z) + \lim_{s \rightarrow t_i^-} u(\gamma(s))$. Similarly we can analyze the case

$$(31) \quad \liminf_{s \rightarrow t_i^-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} \geq 0 \quad \text{and} \quad \limsup_{s \rightarrow t_i^-} \frac{(\gamma(s) - \gamma(t)) \cdot \mathbf{n}}{s - t} > 0.$$

and the behavior of γ at t_i from the right (if $t_i < \ell$). If we add all contributions to the crossing number, we obtain the required equality

$$u(\gamma(\ell)) - u(\gamma(0)) = - \otimes (\varphi, \gamma).$$

In particular, if neither $\gamma(\ell)$ nor $\gamma(0)$ belong to $\varphi(E)$, then the increment is integer. By Proposition 1, almost every line segment P in Ω parallel to a coordinate axis omits the exceptional set $A_1 \cup A_2 \cup A_3$ from Section 3 and has the property that $u|_{P \setminus \varphi(E)}$ attains only finite number of values whose mutual differences are integer. Define the function $v: \Omega \rightarrow \mathbb{C}$ by

$$v(x) = e^{2\pi i u(x)}, \quad x \in \Omega.$$

Then along a.e. line parallel to a coordinate axis, v is a.e. constant. Therefore, the partial derivatives of v in the sense of distributions vanish, which in turn implies that v is constant. It follows that, up to an additive constant C , u is a.e. integer valued and we may assume that $C = 0$.

(ii) $\implies$ (iii) is trivial.

(iii) $\implies$ (i). We may assume that Ω is connected. Let γ be a smooth closed curve in Ω . Find $\delta > 0$ such that

$$(32) \quad \langle \gamma \rangle \subset \Omega_\delta := \{x \in \Omega : \text{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}.$$

We define a mapping $\Phi : E \times [0, \ell(\gamma)] \rightarrow \mathbb{R}^n$ as $\Phi(y, t) = \gamma(t) - \varphi(y)$. Then for a.e. $y \in E$

$$J\Phi(y, t) = \det(-D_1\varphi(y), \dots, -D_{n-1}\varphi(y), \gamma'(t)) = \gamma'(t) \cdot N_\varphi(y).$$

Note that

$$\sum_{\{(y,t) : \gamma(t) - \varphi(y) = 0\}} \text{sgn } J\Phi(y, t) = \otimes(\varphi, \gamma)$$

whenever $\gamma \in \Gamma_\varphi$ and thus

$$(33) \quad \sum_{\{(y,t) : \Phi(t,y) = x\}} \text{sgn } J\Phi(y, t) = \otimes(\varphi, \gamma - x)$$

for a.e. $x \in B(0, \delta)$. Let $(\omega_\delta)_{\delta>0}$ be a standard family of mollifiers in $\mathbb{R}^n$ so that each ω_δ is supported in $B(0, \delta)$. For $x \in B(0, \delta)$ we have

$$(\omega_\delta * \vec{\nu}_\varphi)(x) = \int_\Omega \omega_\delta(x - x') d\vec{\nu}_\varphi(x') = \int_E \omega_\delta(x - \varphi(y)) N_\varphi(y) dy.$$

Using the area formula [40, Section 3.2.3], (33) and Proposition 1 we obtain

$$\begin{aligned} (34) \quad & \int_\gamma (\omega_\delta * \vec{\nu}_\varphi)(x) d\tau_\gamma(x) = \int_0^\ell (\omega_\delta * \vec{\nu}_\varphi)(\gamma(t)) \cdot \gamma'(t) dt \\ & = \int_{E \times [0, \ell]} \omega_\delta(\gamma(t) - \varphi(y)) N_\varphi(y) \cdot \gamma'(t) dy dt \\ & = \int_{E \times [0, \ell]} \omega_\delta(\gamma(t) - \varphi(y)) J\Phi(y, t) dy dt = \int_{B(0, \delta)} \otimes(\varphi, \gamma - x) \omega_\delta(x) dx = 0. \end{aligned}$$

Here we used the assumption (iii).

Now, (34) holds for each smooth closed curve in Ω_δ and using Proposition 2 we obtain that the measure $\omega_\delta * \vec{\nu}_\varphi$ represents a gradient of a function $w_\delta \in C^1(\Omega_\delta)$ if δ satisfies (32).

Choose open connected sets $\Omega_1 \subset \subset \Omega_2 \subset \subset \dots \subset \subset \Omega$ with Lipschitz boundaries such that

$$\bigcup_i \Omega_i = \Omega.$$

Subtracting suitable constants from w_δ , for δ small enough we obtain $v_{i,\delta}$ with zero mean value in Ω_i such that $Dv_{i,\delta} = -\omega_\delta * \vec{\nu}_\varphi$ in Ω_i . In addition to the uniform bound of $Dv_{i,\delta}$ in $L^1(\Omega_i)$, the Poincaré inequality [50, Theorem 12.23] gives a uniform bound for $v_{i,\delta}$ in $L^1(\Omega_i)$. Passing to the limit as $\delta \rightarrow 0$, we obtain, by the compact embedding theorem [32, Theorem 2.5.1], [35, Theorem 3.23], in each Ω_i a function $v_i \in BV(\Omega_i)$ with $Dv_i = -\vec{\nu}_\varphi$. For $j > i$, $Dv_j = Dv_i$ in Ω_i and thus $v_j - v_i$ is constant in Ω_i . Adjusting the functions v_i accordingly we obtain functions $u_i \in BV(\Omega_i)$ such that $u_j = u_i$ in Ω_i for $j > i$ and $Du_i = -\vec{\nu}_\varphi$. The functions u_i define the required function $u \in BV_{\text{loc}}(\Omega)$. $\square$

Example 1. Let $E \subset \mathbb{R}^2$ and $\varphi : E \rightarrow \mathbb{R}^3$ be the mapping of spherical coordinates:

$$\varphi(y) = (\cos y_2 \cos y_1, \cos y_2 \sin y_1, \sin y_2).$$

If $E = (0, 2k\pi) \times (-\pi/2, \pi/2)$, then φ parametrizes the boundary of the unit ball B with $\theta = k$ on ∂B (see (5) for the notation) up to a set Z of $\mathcal{H}^{n-1}(Z) = 0$, which is negligible for our purpose. Observe that (1) holds with $u = k$ on B and $u = 0$ outside B .

If $E = (0, 2\kappa\pi) \times (-\pi/2, \pi/2)$ with $\kappa > 0$ noninteger, then the properties (i)–(iii) of Theorem 1 fail.

Example 2. Let $E_1 = (-4\pi, -2\pi)$, $E_2 = (2\pi, 4\pi)$, $E = E_1 \cup E_2$. Choose $a, b \in \mathbb{R}^2$ and $r, s > 0$. Set

$$\varphi(y) = \begin{cases} (a_1 + r \cos y, a_2 + r \sin y), & y \in E_1, \\ (b_1 + s \cos y, b_2 + s \sin y), & y \in E_2. \end{cases}$$

If either $a \neq b$ or $r \neq s$, then the multiplicity θ of the surface $\varphi(E)$ is 1 $\mathcal{H}^1$ -a.e., but the multiplicity u of (1) may attain all values 0, 1, 2 (if the surrounded discs intersect and both circles are positively oriented) or all values 0, -1 , 1 (if the surrounded discs intersect and the circles are oppositely oriented). It follows that the fact that $\theta = 1$ does not simplify the situation with u .

(On the other hand, if u happens to be the characteristic function of set $A \subset \mathbb{R}^n$, then θ must be 1 a.e. on $\varphi(E)$. Note however, that it can occur even if the multiplicity $\{y \in E: \varphi(y) = x\}$ is different from 1 on a set of positive $\mathcal{H}^{n-1}$ -measure; consider the current example with $a = b$, $r = s$, opposite orientation and $A = \emptyset$.)

7. APPLICATION TO MANIFOLDS

We recall some concepts of theory of manifolds. For details we refer to textbooks on differentiable manifolds, e.g. [30], [49].

Let $\mathcal{M}$ be a separable metrizable topological space. An n -dimensional coordinate map is a homeomorphism $\varkappa: U \rightarrow \mathbb{R}^n$ where $U \subset \mathcal{M}$ is open and connected. The pair $(U, \varkappa)$ is called an n -dimensional coordinate system. A collection $\mathcal{A}$ of n -dimensional coordinate systems $\{(U_\alpha, \varkappa_\alpha): \alpha \in I\}$ is called an n -dimensional $\mathcal{C}^1$ atlas if $\bigcup_{\alpha \in I} U_\alpha = \mathcal{M}$ and each the mapping $\varkappa_\alpha \circ \varkappa_\beta^{-1}$ is $\mathcal{C}^1$ on $\varkappa_\beta(U_\alpha \cap U_\beta)$. We say that the atlas $\mathcal{C}^1$ is oriented if for each $\alpha, \beta \in I$ we have

$$\det \nabla(\varkappa_\alpha \circ \varkappa_\beta^{-1}) > 0 \text{ on } \varkappa_\beta(U_\alpha \cap U_\beta).$$

Let $\mathcal{M}$ be an oriented n -dimensional $\mathcal{C}^1$ manifold. Then there exists an oriented n -dimensional $\mathcal{C}^1$ atlas $\mathcal{A}$ on $\mathcal{M}$ such that all coordinate maps $\varkappa_\alpha$, $\alpha \in I$, are orientation preserving. Conversely, if $\mathcal{A}$ is an oriented n -dimensional $\mathcal{C}^1$ atlas $\mathcal{A}$ on a separable metrizable topological space $\mathcal{M}$, then $\mathcal{A}$ induces a unique structure on an n -dimensional oriented $\mathcal{C}^1$ manifold on $\mathcal{M}$.

Let $\varkappa: U \rightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ -coordinate map. For $i \in \{1, \dots, n\}$ and $x \in U$, we define the functional

$$\left. \frac{\partial}{\partial x_i} \right|_x : f \mapsto D_i(f \circ \varkappa^{-1})(\varkappa(x)), \quad f \in \mathcal{C}_c^1(\mathcal{M}).$$

The tangent space at x is defined as the linear span of the functionals $\left. \frac{\partial}{\partial x_i} \right|_x$, $i = 1, \dots, n$; it does not depend on the choice of the coordinate system.

We consider a Riemannian metrics g on $\mathcal{M}$, this means that with each $x \in \mathcal{M}$ we associate an inner product $g_x(\cdot, \cdot)$ on the tangent space $T_x(\mathcal{M})$ such that the metrics coordinates

$$g_{ij}(x) = g_x\left(\left. \frac{\partial}{\partial x_i} \right|_x, \left. \frac{\partial}{\partial x_j} \right|_x\right)$$

are continuous for all $i, j \in \{1, \dots, n\}$ and each coordinate map $\varkappa_\alpha$, $\alpha \in I$. The norm of a tangent vector $\vec{v} \in T_x(\mathcal{M})$ is $\|\vec{v}\| = \sqrt{g_x(\vec{v}, \vec{v})}$.

Let $\gamma: [a, b] \rightarrow \mathcal{M}$ be a curve. Let $t \in (a, b)$ and $x = \gamma(t)$. If all the composed functions $f \circ \gamma$, $f \in \mathcal{C}_c^1(\mathcal{M})$, are differentiable at t , we define the derivative $\gamma'(t) \in T_x(\mathcal{M})$ as the functional

$$f \mapsto (f \circ \gamma)'(t), \quad f \in \mathcal{C}_c^1(\mathcal{M})$$

The element of length of γ is then $ds = \|\gamma'(t)\| dt$. We say that γ is a path if the all composition $\varkappa_\alpha \circ \gamma$, $\alpha \in I$, are Lipschitz and γ is parametrized by the arclength computed through integration ds . The structure of metric space on $\mathcal{M}$ can be introduced through the geodesic distance, which is locally equivalent to $(x, y) \mapsto |\varkappa_\alpha(x) - \varkappa_\alpha(y)|$ if $x, y \in U_\alpha$, $\alpha \in I$.

We define a Lipschitz surface as a Lipschitz mapping $\varphi: E \rightarrow \mathcal{M}$ where $E \subset \mathbb{R}^{n-1}$ is a measurable set such that $0 < |E| < \infty$. Then integrals of differential $(n-1)$ forms on $\mathcal{M}$ over φ are defined in an obvious way: a differential $(n-1)$ form ω can be locally expressed in coordinates $\varkappa = (x_1, \dots, x_n)$ as a sum

$$\omega = \omega_1 dx_2 \wedge \dots \wedge dx_n + \dots + (-1)^{n-1} \omega_n dx_1 \wedge \dots \wedge dx_{n-1}.$$

Then

$$\begin{aligned} \int_\varphi \omega &= \int_E \left(\omega_1(\varphi(y)) \det \left(\frac{\partial x_i}{\partial y_j}(\varphi(y)) \right)_{\substack{i=2, \dots, n \\ j=1, \dots, n-1}} \right. \\ &\quad \left. + \dots + (-1)^{n-1} \omega_n(\varphi(y)) \det \left(\frac{\partial x_i}{\partial y_j}(\varphi(y)) \right)_{\substack{i=1, \dots, n-1 \\ j=1, \dots, n-1}} \right) dy. \end{aligned}$$

The integrals of n -form over $\mathcal{M}$ are defined as usually through the coordinate maps. In both cases we need to use a partition of unity to complete the definition.

We say that a function $u: \mathcal{M} \rightarrow \mathbb{R}$ is *locally BV* if $u \circ \varkappa_\alpha^{-1}$ is locally *BV* on its domain for each $\alpha \in I$.

The Riemannian structure together with the orientation determines a volume form and thus a reference measure $\mathbf{m}$ for integration of functions over $\mathcal{M}$. Therefore we may introduce *AM*-modulus for path families with values in $\mathcal{M}$ like in the Euclidean setting.

In the rest of the paper we consider a connected oriented n -dimensional Riemannian $\mathcal{C}^1$ manifold $\mathcal{M}$ whose orientation is given by an oriented n -dimensional $\mathcal{C}^1$ atlas $\mathcal{A}$ on $\mathcal{M}$.

Let $\varphi: E \rightarrow \mathcal{M}$ be an $(n-1)$ -dimensional parametric surface and $\gamma: [0, \ell] \rightarrow \mathcal{M}$ be a path (parametrized by arclength). Let $(y_0, t_0) \in E \times [0, \ell]$ and assume that $\gamma(t_0) = \varphi(y_0) = x_0 \in \mathcal{M}$. Find $\alpha \in I$ such that $x_0 \in U_\alpha$. Set

$$(35) \quad E_\alpha = \{y \in E : \varphi(y) \in U_\alpha\}, \quad \varphi_\alpha(y) = \varkappa_\alpha(\varphi(y)), \quad y \in E_\alpha.$$

Further find a path $\gamma_\alpha: [0, \ell_\alpha] \rightarrow \mathbb{R}^n$ and an increasing function $\psi: [0, \ell_\alpha] \rightarrow [0, \ell]$ such that $t_0 \in \psi([0, \ell_\alpha])$, $\gamma(\psi([0, \ell_\alpha])) \subset U_\alpha$ and

$$\gamma_\alpha(t) = \varkappa_\alpha(\gamma(\psi(t))), \quad t \in [0, \ell_\alpha].$$

This selects a subpath of γ which can be covered by one coordinate map. The purpose of the reparametrizing function ψ is to enable arclength parametrization of both γ and γ_α . We say that (y_0, t_0) is a couple of *outer exit* for (φ, γ) if $(y_0, \psi^{-1}(t_0))$ is a couple of outer exit for $(\varphi_\alpha, \gamma_\alpha)$. The set of all couples of outer exit for (φ, γ) is denoted by $C^+(\varphi, \gamma)$. Similarly we define $C^-(\varphi, \gamma)$, $C_+(\varphi, \gamma)$ and $C_-(\varphi, \gamma)$. It is evident that these definitions do not depend on the choice of α and ψ . Now, we may define the *crossing number* $\otimes(\varphi, \gamma)$ by (9).

Our aim is to establish the following theorem; the proof is postponed to Section 8.

Theorem 7. *Let $\varphi: E \rightarrow \mathcal{M}$ be a locally Lipschitz surface. The following statements are equivalent:*

- (i) *There exists a locally BV function $u: \mathcal{M} \rightarrow \mathbb{R}$ such that $\int_\varphi \omega = \int_{\mathcal{M}} u d\omega$ for each $\mathcal{C}^1$ differential $(n-1)$ -form ω with compact support in $\mathcal{M}$,*
- (ii) *There exists a locally BV function $u: \mathcal{M} \rightarrow \mathbb{R}$ such that $u(x) \in \mathbb{N}$ for a.e. x and*

$$u(\gamma(\ell)) - u(\gamma(0)) = - \otimes(\varphi, \gamma)$$

for AM-a.e. path $\gamma: [0, \ell] \rightarrow \mathcal{M}$.

(iii) $\otimes(\varphi, \gamma) = 0$ for AM -a.e. closed path $\gamma: [0, \ell] \rightarrow \mathcal{M}$.

The following examples show how the situation is simple in $\mathbb{R}^2$, but, on the contrary, turns to be complicated in two-dimensional manifolds, in fact, even in open subsets of $\mathbb{R}^2$. We conclude, that the criterion of “closed curve”, or, more sophisticatedly, “boundaryless current” fails on some manifolds not homeomorphic to the entire plane, which can motivate to use our more reliable criterion of “crossing number” instead.

Example 3. Let $\mathcal{M} = \mathbb{R}^2$, $E = (a, b)$ and $\varphi: (a, b) \rightarrow \mathbb{R}^2$ be a Lipschitz curve. Then the following assertions are equivalent:

- (i) There exists $u \in BV(\mathcal{M})$ such that $Du = -\vec{\nu}_\varphi$,
- (ii) $\varphi(b-) = \varphi(a+)$ (φ is closed),
- (iii) The boundary of $\vec{\nu}_\varphi$ in the sense of currents in $\mathcal{M}$ vanishes.

Remark 5. The statement that “the boundary of $\vec{\nu}_\varphi$ in the sense of currents in $\mathcal{M}$ vanishes” means in this situation that

$$\int_{\varphi} d\omega = 0$$

for each smooth differential 0-form ω with compact support in $\mathcal{M}$. In the real-analytic language this means that for any smooth function η with compact support in $\mathbb{R}^2$ we have

$$\int_{\mathbb{R}^2} \det(\nabla \eta, d\vec{\nu}_\varphi) = 0.$$

Example 4. Let $\mathcal{M}$ be the open unit ball in $\mathbb{R}^2$, $E = (-1, 1)$ and $\varphi(y) = (y, 0)$, $y \in E$. Then the conditions (i), (iii) of Example 3 holds but (ii) fails ($\langle \varphi \rangle$ is not relatively compact in $\mathcal{M}$).

Example 5. Let $\mathcal{M} = \{x \in \mathbb{R}^2: 1 < x_1^2 + x_2^2 < 2\}$, $E = (1, 2)$, $\varphi(y) = (y, 0)$, $y \in E$. Then the condition (iii) of Example 3 holds but (i), (ii) fail ($\mathcal{M}$ is not simply connected).

Example 6. Let $\mathcal{M}$ be a 2-dimensional anuloid. We can imagine it as embedded into $\mathbb{R}^3$:

$$\mathcal{M} = \left\{ x \in \mathbb{R}^3: (\sqrt{x_1^2 + x_2^2} - 2)^2 + x_3^2 = 1 \right\}.$$

Let $E = [a, b]$ and $\varphi: E \rightarrow \mathcal{M}$ be a closed Jordan curve (so that (ii) and (iii) of Example 3 are satisfied). If all $\varphi(E)$ is contained in a domain U of a coordinate map $\varkappa$ such that $\varkappa(U)$ is convex, then φ represents a boundary in the sense of Theorem 7(i). However, it can happen that φ does not represent a boundary, for example,

$$\varphi(y) = (\cos y, \sin y, 0), \quad y \in [0, 2\pi].$$

8. PROOF OF THE MAIN THEOREM: MANIFOLD SETTING

Lemma 4. Let $B \subset \mathbb{R}^n$ be a ball and (ρ_j) be a sequence of nonnegative measurable functions on B such that

$$\liminf_{j \rightarrow 0} \int_B \rho_j(z) dz < \infty.$$

Then for a.e. $(x, y) \in B \times B$ there exists a line segment $\gamma_{x,y}$ in B connecting x and y and satisfying

$$(36) \quad \liminf_{j \rightarrow 0} \int_{\gamma_{x,y}} \rho_j ds < \infty.$$

Proof. We may assume that $B = B(0, 1)$. We use the Fatou lemma, the Fubini theorem and the changes of variables first from y to ξ , $\xi = y - x$, and then from x to z , $z = x + t\xi$, for the following estimation. Note that the Jacobians of these transformations are equal to 1.

$$\begin{aligned}
& \int_{B \times B} \liminf_{j \rightarrow \infty} \left(\int_0^1 \rho_j(x + t(y - x)) dt \right) dx dy \\
& \leq \liminf_{j \rightarrow \infty} \int_{B \times B} \left(\int_0^1 \rho_j(x + t(y - x)) dt \right) dx dy \\
& \leq \liminf_{j \rightarrow \infty} \int_0^1 \left(\int_B \int_{B(0,2)} \rho_j(x + t\xi) d\xi dx \right) dt \\
& = \liminf_{j \rightarrow \infty} \int_0^1 \left(\int_{B(0,2)} \int_B \rho_j(x + t\xi) dx d\xi \right) dt \\
& \leq \liminf_{j \rightarrow \infty} \int_0^1 \left(\int_{B(0,2)} \int_B \rho_j(z) dz d\xi \right) dt = |B(0, 2)| \liminf_{j \rightarrow \infty} \int_B \rho_j dz < \infty.
\end{aligned}$$

This means that there exists a set $Z \subset B \times B$ such that each $(x, y) \in (B \times B) \setminus Z$ has the property that the segment $\gamma_{x,y}$ from x to y satisfies (36). $\square$

Lemma 5. *Let Γ be a family of paths in $\mathcal{M}$ such that $AM(\Gamma) = 0$. Then there exists a set $N \subset \mathcal{M}$ of measure zero such that each $x \in \mathcal{M} \setminus N$ and $y \in \mathcal{M} \setminus N$ can be connected by a path not in Γ .*

Proof. Let (ρ_j) be an admissible sequence for Γ such that

$$\liminf_{j \rightarrow \infty} \int_{\mathcal{M}} \rho_j d\mathbf{m} < \infty$$

and

$$\liminf_{j \rightarrow \infty} \int_{\gamma} \rho_j ds = \infty, \quad \gamma \in \Gamma.$$

The existence of such a sequence follows from [42, Theorem 7]. Define

$$\bar{\Gamma} = \left\{ \gamma : \liminf_j \int_{\gamma} \rho_j ds = \infty \right\}.$$

and note that $\Gamma \subset \bar{\Gamma}$. By Lemma 4, for each point $x_0 \in \mathcal{M}$ there exists its neighborhood U and a set $Z \subset U \times U$ such that $(\mathbf{m} \otimes \mathbf{m})(Z) = 0$ (the symbol just used denotes the product measure) and for each $(x, y) \in (U \times U) \setminus Z$ there exists a path $\gamma \notin \bar{\Gamma}$ connecting x and y in U . Let us consider U , Z as above. By the Fubini theorem, there exist N_1, N_2 of measure zero such that for each $x' \in U \setminus N_1$ we have $\mathbf{m}(\{y \in U : (x', y) \in Z\}) = 0$ and for each $y' \in U \setminus N_2$ we have $\mathbf{m}(\{x \in U : (x, y') \in Z\}) = 0$.

Fix $x' \in U \setminus N_1$ and find $y' \in U \setminus N_2$ such that $(x', y') \notin Z$. Then there exist sets $N_3, N_4 \subset U$ of measure zero such that for each $x \in U \setminus N_3$ we have $(x, y') \notin Z$ and for each $y \notin N_4$ we have $(x', y) \notin Z$. Set $N_U = N_3 \cup N_4$, then each $x, y \in U \setminus N_U$ can be connected by a path not in $\bar{\Gamma}$ so that we merge the paths $\gamma_{x,y'}$, $\gamma_{y',x'}$ and $\gamma_{x',y}$.

Fix U, N_U as above and define $\mathcal{P}$ as the set of all $z \in \mathcal{M}$ with the property that there is a neighborhood V of z and a set $N_V \subset V$ of measure zero such that each $x \in U \setminus N_U$ and $y \in V \setminus N_V$ can be connected by a path $\gamma_{x,y} \notin \bar{\Gamma}$. The set $\mathcal{P}$ is nonempty as it contains U ; it is obviously open. Let us prove that $\mathcal{P}$ is closed. Fix $z_0 \in \bar{\mathcal{P}}$. As above, find a neighborhood W of z_0 and a set N_0 of measure zero such that for each $z, y \in W \setminus N_0$ there is $\gamma_{z,y} \notin \bar{\Gamma}$ connecting z and y . Then find $z \in \mathcal{P} \cap W$ and a neighborhood $V_1 \subset W$ of z and a set $N_1 \subset V_1$ of measure zero

such that each $x \in U \setminus N_U$ and $y_1 \in V_1 \setminus N_1$ can be connected by a path $\gamma_{x,y_1} \notin \bar{\Gamma}$. Find $y_1 \in V_1 \setminus (N_1 \cup N_0)$. If $x \in U \setminus N_U$ and $y \in W \setminus N_0$, then we can connect x with y through y_1 . Therefore $z_0 \in P$. We have verified that $\mathcal{P}$ is both open and closed and from connectedness of $\mathcal{M}$ we infer that $\mathcal{P} = \mathcal{M}$. Finally, since $\mathcal{M}$ has a countable basis of its topology, we conclude that there is a single set N of measure zero (not depending on z_0) such that each $x, y \in \mathcal{M} \setminus N$ can be connected by a path $\gamma_{x,y} \notin \bar{\Gamma}$. $\square$

Proof of Theorem 7. (i) $\implies$ (ii). Let $(U_\alpha, \kappa_\alpha) \in \mathcal{A}$ and ω have a support in U_α . Denote by $x_1, \dots, x_n$ the coordinates of $\kappa = \kappa_\alpha$ and $\Omega = \kappa_\alpha(U_\alpha)$. If $\varphi_\alpha: E_\alpha \rightarrow \mathbb{R}^n$ is as (35), then there exists a function $f = (f_1, \dots, f_n) \in \mathcal{C}_c^1(\Omega)$ such that

$$\omega = (f_1 \circ \kappa^{-1}) dx_2 \wedge \dots \wedge dx_n + \dots + (-1)^{n-1} (f_n \circ \kappa^{-1}) dx_1 \wedge \dots \wedge dx_{n-1}$$

and

$$d\omega = (\operatorname{div} f \circ \kappa^{-1}) dx_1 \wedge \dots \wedge dx_n.$$

Then the condition (i) means that

$$D(u \circ \kappa_\alpha^{-1}) = -\vec{\nu}_{\varphi_\alpha} \quad \text{in } \Omega$$

and Theorem 1(i) $\implies$ (ii) shows that the property (ii) holds for “short paths”, namely paths whose locus is contained in some U_α . The general case follows by partitioning a general path into short subpaths.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (ii). We use the Euclidean version of the implication to find functions u_α which are BV in U_α and satisfy (ii) in U_α . Let Γ_1 be the family of all paths which contain a subpath γ with locus in some U_α and violating the formula

$$u_\alpha(\gamma(\ell)) - u_\alpha(\gamma(0)) = -\otimes(\varphi, \gamma).$$

Further, let Γ_2 be the family of all closed paths which violate

$$\otimes(\varphi, \gamma) = 0.$$

Then $AM(\Gamma_2) = 0$ by (iii) and $AM(\Gamma_1) = 0$ by the Euclidean version of the implication (iii) $\implies$ (ii). Set $\Gamma = \Gamma_1 \cup \Gamma_2$. Find an admissible sequence (ρ_j) for Γ such that

$$\liminf_{j \rightarrow \infty} \int_{\mathcal{M}} \rho_j d\mathbf{m} < \infty$$

and

$$\liminf_{j \rightarrow \infty} \int_{\gamma} \rho_j ds = \infty, \quad \gamma \in \Gamma.$$

The existence of such a sequence follows from [42, Theorem 7]. Set

$$\bar{\Gamma} = \left\{ \gamma: \liminf_j \int_{\gamma} \rho_j ds = \infty \right\}.$$

and note that $\Gamma \subset \bar{\Gamma}$ and $AM(\bar{\Gamma}) = 0$. By Lemma 5 there exists a set $N \subset \mathcal{M}$ of measure zero such that each $x \in \mathcal{M} \setminus N$ and $y \in \mathcal{M} \setminus N$ can be connected by a path not in $\bar{\Gamma}$. Fix $x_0 \in \mathcal{M} \setminus N$ and define the function u first on $\mathcal{M} \setminus N$ as

$$u(x) = \otimes(\varphi, \gamma), \quad x \in \mathcal{M} \setminus N$$

where $\gamma \notin \bar{\Gamma}$ is a path connecting x_0 and x . This definition does not depend on the choice of γ : if γ_1 and $\gamma_2 \notin \bar{\Gamma}$ are paths connecting x_0 and x , then we can merge γ_1 and the oppositely parametrized γ_2 to obtain a closed path $\gamma \notin \bar{\Gamma}$, so that

$$\otimes(\varphi, \gamma_1) - \otimes(\varphi, \gamma_2) = \otimes(\varphi, \gamma) = 0$$

For any α we find c_α such that $u = u_\alpha + c_\alpha$ on $U_\alpha \setminus N$. Let us define u as $u_\alpha + c_\alpha$ on U_α . Obviously, this definition does not depend on α regarding the values of u on some $U_{\alpha_1} \cap U_{\alpha_2}$.

Now, note that for AM -a.e. path $\gamma: [0, \ell] \rightarrow \mathcal{M}$ we have $|\{t: \gamma(t) \in N\}| = 0$. This follows by using an admissible function ρ with $\rho = \infty$ on N .

If $\gamma: [0, \ell] \rightarrow \mathcal{M}$ does not belong to $\bar{\Gamma}$ and satisfies $|\{t: \gamma(t) \in N\}| = 0$, then we find $a, b \in (0, \ell)$ and α, β such that $\gamma(t) \in U_\alpha$ from $t \in [0, a]$, $\gamma(t) \in U_\beta$ for $t \in [b, \ell]$ and $\gamma(a), \gamma(b) \notin N$. Then the crossing formula for γ holds on $[0, a]$, $[a, b]$ and $[b, \ell]$, so that it holds on $[0, \ell]$.

(ii) $\implies$ (i): As in the proof of (i) $\implies$ (ii) we realize that local validity of the Stokes formula (i) in U_α is equivalent to the validity of the divergence formula (1) in $\kappa(U_\alpha)$. The Stokes formula (i) holds with the multiplicity u locally, so by means of a suitable partition of unity we can verify that it holds globally as well. $\square$

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