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STRUCTURAL RIGIDITY OF GENERALISED VOLTERRA OPERATORS ON H^p

SANTERI MIIHKINEN, PEKKA J. NIEMINEN, EERO SAKSMAN, AND HANS-OLAV TYLLI

ABSTRACT. We show that the non-compact generalised analytic Volterra operators T_g , where $g \in BMOA$, have the following structural rigidity property on the Hardy spaces H^p for $1 \leq p < \infty$ and $p \neq 2$: if T_g is bounded below on an infinite-dimensional subspace $M \subset H^p$, then M contains a subspace linearly isomorphic to ℓ^p . This implies in particular that any Volterra operator $T_g: H^p \rightarrow H^p$ is ℓ^2 -singular for $p \neq 2$.

1. INTRODUCTION

In this paper we establish some structural rigidity properties of the generalised Volterra-type integral operators

$$f \mapsto T_g f(z) = \int_0^z f(w)g'(w) dw, \quad z \in \mathbb{D},$$

on the Hardy spaces H^p for $p \neq 2$, where g is a given analytic function defined on the open unit disc $\mathbb{D}$ of the complex plane. Various properties of this class of operators have been investigated since the mid-1990s on H^p , as well as on many other function spaces; see e.g. the surveys [2] and [22]. In particular, T_g is bounded (respectively, compact) $H^p \rightarrow H^p$ if and only if $g \in BMOA$ (respectively, $g \in VMOA$) according to results of Aleman and Siskakis [4] for $1 \leq p < \infty$ and Aleman and Cima [3] for $0 < p < 1$.

Our aim is to study largeness in the linear qualitative sense of the Volterra operators $T_g: H^p \rightarrow H^p$ for $p \neq 2$. We first recall the following general concept for a Banach space X . A bounded operator $U: X \rightarrow X$ is said to fix a copy of the given Banach space E , if there is a closed subspace $M \subset X$, linearly isomorphic to E , and $c > 0$ so that $\|Ux\| \geq c \cdot \|x\|$ for all $x \in M$ (that is, the restriction $U|_M$ defines an isomorphism $M \rightarrow UM$). In this direction, the first author [17] showed that any non-compact operator T_g fixes a copy of ℓ^p in H^p . Our main result demonstrates that the non-compactness of these operators is, in fact, very limited in the qualitative sense.

Theorem 1.1. *Let $g \in BMOA$ be arbitrary and $1 \leq p < \infty$, $p \neq 2$. If the restriction $T_g|_M$ is bounded below on an infinite-dimensional subspace $M \subset H^p$, then M contains a linearly isomorphic copy of ℓ^p .*

We refer to Remark 3.5.(1) below for a more detailed discussion of the contents of Theorem 1.1. For the moment recall only that for $1 \leq q < \infty$ the bounded operator $U: X \rightarrow X$ is called ℓ^q -singular, denoted $U \in \mathcal{S}_q(X)$, if U does not fix any copies of ℓ^q in X . The following corollary is an immediate consequence of Theorem 1.1 and the fact that the sequence spaces ℓ^p and ℓ^2 are totally incomparable for $p \neq 2$, see e.g. [16, 2.a.1 and p. 54]. It also answers a question from [17, Sec. 3] in the negative. Note further that Theorem 1.1 is obvious for H^2 , while the corollary fails for $p = 2$.

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Corollary 1.2. *Let $1 \leq p < \infty$ and $p \neq 2$. Then $T_g \in \mathcal{S}_2(H^p)$ for every $g \in BMOA$, that is, T_g does not fix any linearly isomorphic copies of ℓ^2 in H^p .*

In particular, $T_g: H^p \rightarrow H^p$ cannot fix a copy of the whole space H^p itself in the case $p \neq 2$.

It is of interest to contrast the preceding results with the behaviour of the composition operators $f \mapsto C_\phi f = f \circ \phi$ where $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is any fixed analytic map. It is well known that C_ϕ is a bounded operator $H^p \rightarrow H^p$, cf. [7, Thm 3.1]. It was recently shown [14] that for $p \neq 2$ any non-compact $C_\phi: H^p \rightarrow H^p$ fixes an isomorphic copy of ℓ^p in H^p (akin to the case of the Volterra operators). However, there are many examples of composition operators C_ϕ that fix a copy of ℓ^2 in H^p for $p \neq 2$ and this class can be characterised in terms of the boundary behaviour of the symbol ϕ , see [14, Thm 1.4].

Theorem 1.1 will be proved in Section 3. In Section 2 we sketch as an appetiser an argument for the ℓ^2 -singularity of the Cesàro operator on H^p , or equivalently, the ℓ^2 -singularity of the non-compact Volterra operator T_h obtained with the symbol $h(z) = \log \frac{1}{1-z} \notin VMOA$. We consider it worthwhile to look at this classical example separately, since it exemplifies a “localisation” of the non-compactness, which somewhat surprisingly turns out to hold for arbitrary symbols g in $BMOA$ in a suitable formulation.

We refer to [8], [9] and [11] for unexplained notions and results for the spaces H^p and $BMOA$, and respectively [16] and [1] for general Banach space theory. We recall here only that the analytic map $f: \mathbb{D} \rightarrow \mathbb{C}$ belongs to $BMOA$ if $|f|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ for $z \in \mathbb{D}$, and that $BMOA$ is a Banach space equipped with the norm $\|f\| = |f(0)| + |f|_*$. Moreover, $VMOA$ is the closed subspace of $BMOA$ consisting of the maps $f \in BMOA$ for which $\lim_{|a| \rightarrow 1} \|f \circ \sigma_a - f(a)\|_{H^2} = 0$.

2. APPETISER: ℓ^2 -SINGULARITY OF THE CESÀRO OPERATOR

As a prototypical example of a non-compact Volterra-type integral operator we briefly consider the classical Cesàro operator

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w} dw = \frac{1}{z} T_h f(z), \quad z \in \mathbb{D},$$

where $h(z) = \log \frac{1}{1-z}$. Thus $T_h = M_z C$, where the multiplier $f \mapsto M_z f = zf$ is an isometry on H^p . A systematic study of C on H^p was initiated by Siskakis; see e.g. [20] and [21].

Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. Observe that $h'(z) = \frac{1}{1-z}$ is bounded on $\mathbb{T} \setminus J$, where J is any open arc containing the point $z = 1$, and its values peak at the point $z = 1$. Intuitively speaking, C behaves like a compact operator on a major part of $\mathbb{T}$ and in a non-compact fashion near $z = 1$. It is known by [17, Thm 1.1] that C fixes a copy of ℓ^p . We next outline a proof of the following special case of Corollary 1.2.

Claim. *The Cesàro operator C is ℓ^2 -singular on H^p for $1 < p < \infty$ and $p \neq 2$.*

Proof. Assume to the contrary that C fixes a copy of ℓ^2 . Thus there is a sequence (f_n) in H^p which is equivalent to the unit vector basis of ℓ^2 , and such that C is bounded below on the closed linear span of (f_n) . Put $E_\varepsilon = \{e^{it} : |e^{it} - 1| < \varepsilon\}$ for $\varepsilon > 0$. By the absolute continuity of the measures $E \mapsto \int_E |Cf_n|^p dm$, we get that

$$(2.1) \quad \lim_{\varepsilon \rightarrow 0} \int_{E_\varepsilon} |Cf_n|^p dm = 0$$

for each n . We next claim that

$$(2.2) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{T} \setminus E_\varepsilon} |Cf_n|^p dm = 0$$

for each $\varepsilon > 0$. In fact, given $0 < \delta < 1$, we have

$$(2.3) \quad |Cf_n(e^{i\theta})| \leq \int_0^{1-\delta} \frac{|f_n(re^{i\theta})|}{|1-re^{i\theta}|} dr + \int_{1-\delta}^1 \frac{|f_n(re^{i\theta})|}{|1-re^{i\theta}|} dr,$$

where the first integral in (2.3) tends to 0 as $n \rightarrow \infty$, since (f_n) converges to zero uniformly on compact subsets of $\mathbb{D}$ (recall that $f_n \rightarrow 0$ weakly in H^p). Moreover, for $e^{i\theta} \notin E_\varepsilon$ the second integral in (2.3) can be made uniformly as small as desired by choosing $\delta > 0$ small enough, and using the estimates $|1-re^{i\theta}| \geq c\varepsilon$ for $\varepsilon \leq |\theta| \leq \pi$ and some $c > 0$, as well as

$$(2.4) \quad |f_n(z)| \leq \frac{2^{1/p} \|f_n\|_{H^p}}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D},$$

for $p \in (0, \infty)$, see [8, p. 36]. Hence (2.2) holds.

Since conditions (2.1) and (2.2) are satisfied and $\|Cf_n\|_{H^p} \approx 1$, we may proceed as in the proof of Proposition 3.5 in [17] (cf. also the proof of Theorem 1.2 in [14]) and extract a subsequence (Cf_{n_k}) which is equivalent to the unit vector basis of ℓ^p in H^p . However, since (Cf_{n_k}) is also equivalent to the unit vector basis of ℓ^2 by assumption, the above contradicts the fact that ℓ^p and ℓ^2 are totally incomparable for $p \neq 2$. This contradiction shows that C is indeed ℓ^2 -singular. $\square$

As noted before, the above concrete operator $T_h \in \mathcal{S}_2(H^p)$ acts like a compact operator on most of $\mathbb{T}$. We now face the problem of establishing a similar behaviour for T_g with an arbitrary symbol $g \in BMOA \setminus VMOA$. In the general case sheer size estimates of the derivative g' will not suffice. For instance, by [19] there is a bounded analytic function $g \in H^\infty$, such that the curves $\{g(re^{it}) : 0 \leq r < 1\}$ are unrectifiable for almost every $e^{it} \in \mathbb{T}$. Instead, to overcome the basic difficulty we will provide a key result for the localisation of non-compactness, see Proposition 3.2 below. Our starting point is a condensation phenomenon (Lemma 3.3) shared by the Carleson measures $|g'(z)|^2 \log \frac{1}{|z|} dA(z)$, which allows us to isolate the “support of the non-compactness” on $\mathbb{T}$. Nonetheless, exploiting this knowledge is not straightforward, and thus we also need to post-compose T_g in Lemma 3.4 by an auxiliary composition operator, whose symbol is geometrically carefully chosen with respect to the “support of the non-compactness.”

3. PROOF OF THEOREM 1.1

We will actually establish the following more precise version of Theorem 1.1.

Theorem 3.1. *Let $g \in BMOA$ be arbitrary and $1 \leq p < \infty$. Suppose that (f_n) is a sequence in H^p such that $\|f_n\|_{H^p} = 1$ for all n and $f_n \rightarrow 0$ uniformly on the compact subsets of $\mathbb{D}$. If*

$$(3.1) \quad \limsup_{n \rightarrow \infty} \|T_g f_n\|_{H^p} > 0,$$

then there is a subsequence (f_{n_j}) such that $(T_g f_{n_j})$ is equivalent to the unit vector basis of ℓ^p , that is, there is constant $c > 0$ for which

$$(3.2) \quad c^{-1} \cdot \left(\sum_j |\alpha_j|^p \right)^{1/p} \leq \left\| \sum_j \alpha_j T_g f_{n_j} \right\|_{H^p} \leq c \cdot \left(\sum_j |\alpha_j|^p \right)^{1/p}$$

holds for all $(\alpha_j) \in \ell^p$.

Assuming Theorem 3.1 for a moment we next derive Theorem 1.1 from it.

Proof of Theorem 1.1. Suppose that the restriction $T_{g|M}$ is bounded below on a closed infinite-dimensional subspace $M \subset H^p$. We may next pick a normalised sequence $(f_n) \subset M$ such that $f_n \rightarrow 0$ uniformly as $n \rightarrow \infty$ on the compact subsets of $\mathbb{D}$. For the reader's convenience we recall that this is obtained by choosing a normalised sequence $(f_n) \subset M$ such that $f_n^{(r)}(0) = 0$ for all $r = 0, \dots, n$ and any $n \geq 1$. The choice is possible since

M is infinite-dimensional and the intersection of the kernels of the evaluation functionals $f \mapsto f^{(r)}(0)$ for $r = 0, \dots, n$ has finite codimension in M . It is then a standard fact that $f_n \rightarrow 0$ as $n \rightarrow \infty$ on the compact subsets of $\mathbb{D}$. In fact, since $f \mapsto z^n f$ is an isometry on H^p , we deduce by (2.4) that for any given $0 < r < 1$ there is a constant $C(r)$ so that for any z satisfying $|z| \leq r$ one has

$$|f_n(z)| \leq C(r)|z|^n \leq C(r)r^n \rightarrow 0$$

as $n \rightarrow \infty$.

After these preparations Theorem 3.1 produces a subsequence (f_{n_j}) for which (3.2) holds for $(T_g f_{n_j})$. Since T_g is bounded below on the closed linear span $[\{f_{n_j} : j \geq 1\}] \subset M$ of (f_{n_j}) , it follows that T_g fixes a linearly isomorphic copy of ℓ^p in M . $\square$

The argument for Theorem 3.1 is based on the following proposition, which shows that the non-compact behaviour of T_g for the symbols $g \in BMOA \setminus VMOA$ is ultimately concentrated on subsets of arbitrarily small measure of the unit circle $\mathbb{T}$. As usual we view H^p as a closed subspace of $L^p(\mathbb{T}) = L^p(\mathbb{T}, m)$, where m is the normalised Lebesgue measure on $\mathbb{T}$.

Proposition 3.2. *Let $g \in BMOA$ and $1 \leq p < \infty$. Then for every $\eta > 0$ there exists a compact set $E \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus E) < \eta$ and $\chi_E T_g : H^p \rightarrow L^p(\mathbb{T})$ is a compact operator.*

In particular, for any bounded sequence $(f_n) \subset H^p$ such that $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ one has $\|(\chi_E T_g) f_n\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$.

Taking Proposition 3.2 for granted momentarily, we next show how it yields Theorem 3.1.

Proof of Theorem 3.1. By a repeated application of Proposition 3.2 we can find compact sets $E_1 \subset E_2 \subset \dots \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus E_k) \rightarrow 0$ as $k \rightarrow \infty$ and $\chi_{E_k} T_g : H^p \rightarrow L^p(\mathbb{T})$ is a compact operator for each k . Moreover, since $f_n \rightarrow 0$ uniformly on the compact subsets of $\mathbb{D}$ by assumption, Proposition 3.2 ensures that

$$(3.3) \quad \|(\chi_{E_k} T_g) f_n\|_{L^p}^p = \int_{E_k} |T_g f_n|^p dm \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for each k . On the other hand, for each n we have by absolute continuity that

$$(3.4) \quad \int_{\mathbb{T} \setminus E_k} |T_g f_n|^p dm \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By assumption (3.1) and after passing to a subsequence we get that $\int_{\mathbb{T}} |T_g f_n|^p dm > d^p$ for each n and a suitable constant $d > 0$. For any fixed $\delta > 0$, we may by successive applications of (3.3) and (3.4) extract increasing indices $n_1 < n_2 < \dots$ and $j_1 < j_2 < \dots$ such that the conditions

$$(3.5) \quad \left(\int_{\mathbb{T} \setminus E_{j_r}} |T_g f_{n_s}|^p dm \right)^{1/p} < 4^{-r} \delta d, \quad s = 1, \dots, r-1,$$

$$(3.6) \quad \left(\int_{E_{j_r}} |T_g f_{n_r}|^p dm \right)^{1/p} < 4^{-r} \delta d$$

$$(3.7) \quad \left(\int_{\mathbb{T} \setminus E_{j_r}} |T_g f_{n_r}|^p dm \right)^{1/p} > d$$

hold for all $r \geq 1$. This is a straightforward gliding hump type argument: Suppose that we have found indices $n_1 < \dots < n_r$ and $j_1 < \dots < j_r$ so that (3.5)–(3.7) hold until r . Next we use (3.4) to get $j_{r+1} > j_r$ so that (3.5) holds for $f_{n_1}, \dots, f_{n_r}$ and the set $\mathbb{T} \setminus E_{j_{r+1}}$, and then (3.3) to find $n_{r+1} > n_r$ so that (3.6) holds. Moreover, we may ensure (3.7) at stage $r+1$ since $\int_{\mathbb{T}} |T_g f_n|^p dm > d^p$ for each n .

Finally, by applying the perturbation argument from [17, Sec. 3] (see also Theorem 1.2 in [14]), which we do not repeat here, it follows that (3.2) holds for $(T_g f_{n_j})$ once $\delta > 0$ is small enough. $\square$

We next turn to the proof of the crucial Proposition 3.2. To any analytic function $g: \mathbb{D} \rightarrow \mathbb{C}$ we associate the positive Borel measure μ_g on $\mathbb{D}$ defined by the density

$$d\mu_g(z) = |g'(z)|^2 \log \frac{1}{|z|} dA(z),$$

where A is the Lebesgue area measure normalised such that $A(\mathbb{D}) = 1$. Then the Littlewood-Paley identity (see e.g. [7, Thm 2.30]) implies that

$$\|g\|_{H^2}^2 = |g(0)|^2 + 2 \int_{\mathbb{D}} d\mu_g(z)$$

for $g \in H^2$. We also recall that $g \in BMOA$ if and only if μ_g is a Carleson measure, i.e. there is a constant $c > 0$ such that

$$\mu_g(W(\zeta, h)) \leq ch \quad \text{for } \zeta \in \mathbb{T}, 0 < h < 1,$$

where $W(\zeta, h)$ is the Carleson window

$$W(\zeta, h) = \{z \in \mathbb{D} : 1 - h < |z| < 1, |\arg(z/\zeta)| < h\}.$$

Furthermore, $g \in VMOA$ if and only if μ_g is a vanishing Carleson measure, i.e.

$$\sup_{\zeta \in \mathbb{T}} \mu_g(W(\zeta, h)) = o(h) \quad \text{as } h \rightarrow 0.$$

For proofs of the above results see e.g. [9, Chap. VI.3] or [11, Sec. 6].

Our first auxiliary result says that every function $g \in BMOA$ (or even $g \in H^2$) has uniformly vanishing mean oscillation on $\mathbb{T}$ up to a set of arbitrarily small measure.

Lemma 3.3. *Let $g \in H^2$ be arbitrary. Then for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus K) < \varepsilon$ and*

$$\sup_{\zeta \in K} \mu_g(W(\zeta, h)) = o(h) \quad \text{as } h \rightarrow 0.$$

Proof. For each $k \geq 1$, let ν_k be the projection to the unit circle of the measure μ_g restricted to the annulus $S_k = \{z : 1 - \frac{1}{k} < |z| < 1\}$. That is, ν_k is determined by the condition

$$\nu_k(I(\zeta, h)) = \mu_g(\{z \in S_k : |\arg(z/\zeta)| < h\})$$

for all boundary arcs $I(\zeta, h) = \{\xi \in \mathbb{T} : |\arg(\xi/\zeta)| < h\}$. Consider the Hardy-Littlewood maximal function of ν_k ,

$$\nu_k^*(\zeta) = \sup_{0 < h < \pi} \frac{1}{2h} \nu_k(I(\zeta, h)).$$

By the maximal function theorem it satisfies

$$(3.8) \quad m(\{\zeta \in \mathbb{T} : \nu_k^*(\zeta) > \lambda\}) \leq C \frac{\nu_k(\mathbb{T})}{\lambda}$$

for all $\lambda > 0$ and a numerical constant $C > 0$. Note that here $\nu_k(\mathbb{T}) = \mu_g(S_k) \rightarrow 0$ as $k \rightarrow \infty$ since μ_g is a finite measure by the Littlewood-Paley identity.

We claim that $\nu_k^* \rightarrow 0$ almost everywhere on $\mathbb{T}$ as $k \rightarrow \infty$. In fact, since (ν_k^*) is obviously a pointwise decreasing sequence, we would otherwise find a constant $\lambda > 0$ and a set $E \subset \mathbb{T}$ of positive measure such that $\nu_k^*(\zeta) > \lambda$ for all $\zeta \in E$ and $k \geq 1$. But this would contradict (3.8). Hence $\nu_k^* \rightarrow 0$ a.e. on $\mathbb{T}$. Egorov's theorem now implies that there is a set $F \subset \mathbb{T}$ with $m(\mathbb{T} \setminus F) < \varepsilon/2$ such that $\nu_k^* \rightarrow 0$ uniformly in F as $k \rightarrow \infty$. Since, for every $k \geq 1$ and $\zeta \in F$,

$$\sup_{0 < h < 1/k} \frac{\mu_g(W(\zeta, h))}{h} \leq \sup_{0 < h < \pi} \frac{\nu_k(I(\zeta, h))}{h} = 2\nu_k^*(\zeta),$$

we deduce that $\sup_{\zeta \in F} \mu_g(W(\zeta, h)) = o(h)$ as $h \rightarrow 0$. Finally, it just remains to pick a compact subset $K \subset F$ with $m(F \setminus K) < \varepsilon/2$. $\square$

To exploit Lemma 3.3 in the analysis of the Volterra operator T_g we will employ the product operator $C_\phi T_g$ for a composition operator C_ϕ whose symbol ϕ will be associated to the compact set K from Lemma 3.3. Here

$$(C_\phi T_g)f(z) = \int_0^{\phi(z)} f(w)g'(w)dw, \quad z \in \mathbb{D},$$

for any analytic function $f: \mathbb{D} \rightarrow \mathbb{C}$. Recall that both C_ϕ and T_g , and hence the product $C_\phi T_g$, are bounded on H^p for all $p \in (0, \infty)$. For each $\zeta \in \mathbb{T}$ define the Stolz domain $S(\zeta)$ in $\mathbb{D}$ with vertex at ζ as the interior of the convex hull of the set $\{z : |z| < \frac{1}{2}\} \cup \{\zeta\}$.

Lemma 3.4. *Let $g \in BMOA$ and $\varepsilon > 0$. Choose a compact subset $K \subset \mathbb{T}$ as in Lemma 3.3 and put $\Omega = \bigcup_{\zeta \in K} S(\zeta)$. Let ϕ be a Riemann map from $\mathbb{D}$ onto Ω with $\phi(0) = 0$. Then $C_\phi T_g: H^p \rightarrow H^p$ is a compact operator for any $1 \leq p < \infty$.*

Proof. We start by considering the case $p = 2$. By the Littlewood-Paley identity, the fact that $\phi(0) = 0$, and the change of variables $w = \phi(z)$, we get that

$$(3.9) \quad \|(C_\phi T_g)f\|_{H^2}^2 = 2 \int_{\Omega} |f(w)|^2 |g'(w)|^2 \log \frac{1}{|\phi^{-1}(w)|} dA(w).$$

Since $|\phi^{-1}(w)| \geq |w|$ for all $w \in \Omega$ by Schwarz's lemma, it will be enough to show that $\chi_\Omega d\mu_g$ is a vanishing Carleson measure. Assuming this for a moment, known results (cf. the proof of [7, Thm 2.33]) yield that the natural embedding $H^2 \rightarrow L^2(\mathbb{D}, \chi_\Omega d\mu_g)$ is a compact operator, whose norm pointwise dominates that of $C_\phi T_g: H^2 \rightarrow H^2$ by (3.9). This easily implies that also $C_\phi T_g: H^2 \rightarrow H^2$ is compact.

In order to verify the vanishing nature of the Carleson measure $\chi_\Omega d\mu_g$, let $\zeta \in \mathbb{T}$ and $0 < h < \frac{1}{4}$, and consider a Carleson window $W(\zeta, h)$ which has a nonempty intersection with Ω . Note that $\partial\Omega \cap \mathbb{T} = K$, and let ξ be a point of K that is closest to ζ (if $\zeta \in K$, then take $\xi = \zeta$). It follows by geometric inspection that the angular distance from ζ to ξ is less than $2h$. Therefore $W(\zeta, h) \subset W(\xi, 3h)$ and so

$$\frac{1}{h}(\chi_\Omega \mu_g)(W(\zeta, h)) \leq 3 \cdot \frac{1}{3h} \mu_g(W(\xi, 3h)).$$

In view of Lemma 3.3 this implies that $\chi_\Omega d\mu_g$ is a vanishing Carleson measure.

The claim for the other values of p is obtained from the case $p = 2$ above and the identification $H^p = (H^{p_0}, H^{p_1})_{\theta, p}$ in terms of real interpolation spaces, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $0 < p_0 < \infty$ and $p_1 = 2$ (see e.g. [12, p. 1]), and one-sided Krasnoselskii-type interpolation of compactness for operators. For the case $p > 1$ the classical form [13] of this result suffices, and for $p = 1$ we refer e.g. to [6, Thm 3.1] for a version of Krasnoselskii's theorem that also applies to quasi-Banach spaces such as H^{p_0} for $0 < p_0 < 1$. $\square$

The auxiliary steps in Lemmas 3.3 and 3.4 finally allow us to establish Proposition 3.2.

Proof of Proposition 3.2. Fix $0 < \eta < 1/2$. Let K , Ω and ϕ be as given by Lemmas 3.3 and 3.4 corresponding to $\varepsilon = \eta/3$. Define a probability measure ν on the closed disc $\overline{\mathbb{D}}$ as the push-forward of m under the boundary values of ϕ , which we denote by $\phi^*(\zeta)$; that is,

$$\nu(B) = m(\{\zeta \in \mathbb{T} : \phi^*(\zeta) \in B\})$$

for all Borel sets $B \subset \overline{\mathbb{D}}$. Then ν is the harmonic measure of the domain Ω with pole at $\phi(0) = 0$. Let h be the density (i.e. the Radon-Nikodym derivative) of $\nu|_{\mathbb{T}}$ (or, equivalently, of $\nu|_K$) with respect to m . It is not difficult to see that $0 \leq h \leq 1$ a.e.

Note that $\partial\Omega$ is rectifiable, since $\Omega \subset \mathbb{D}$ is a Lipschitz domain by inspection. Therefore, according to a classical result of F. and M. Riesz (see Theorem VI.1.2 and (1.4) on p. 202 of [10]), ν and the Hausdorff 1-measure $\mathcal{H}^1$ on $\partial\Omega$ are mutually absolutely continuous. In

particular, if $F \subset K$ satisfies $\nu(F) = 0$, then $m(F) = \mathcal{H}^1(F) = 0$. This implies that $h > 0$ a.e. on K . Thus, we may find $\delta > 0$ small enough so that the set $F = \{\zeta \in K : h(\zeta) > \delta\}$ satisfies $m(K \setminus F) < \varepsilon$. Then we proceed to choose the compact set $E \subset F$ such that $m(F \setminus E) < \varepsilon$, whence $m(\mathbb{T} \setminus E) < 3\varepsilon = \eta$.

With the above preparations we may estimate, for any $f \in H^p$,

$$(3.10) \quad \begin{aligned} \|(C_\phi T_g)f\|_{H^p}^p &= \int_{\mathbb{T}} |(T_g f) \circ \phi^*|^p dm = \int_{\mathbb{D}} |T_g f|^p d\nu \\ &\geq \int_{\mathbb{T}} |T_g f|^p h dm \geq \delta \int_E |T_g f|^p dm. \end{aligned}$$

Since $C_\phi T_g$ is compact $H^p \rightarrow H^p$ by Lemma 3.4, as in the proof of that Lemma, we deduce that $\chi_E T_g: H^p \rightarrow L^p(\mathbb{T})$ is a compact operator.

Towards the final claim of the proposition suppose that the sequence $(f_n) \subset H^p$ is bounded and converges uniformly to 0 on the compact subsets of $\mathbb{D}$. Then also $(C_\phi T_g)f_n \rightarrow 0$ uniformly on the compact subsets of $\mathbb{D}$. Therefore, by the uniqueness of the limit, every norm-convergent subsequence of $((C_\phi T_g)f_n)$ must tend to zero. Since $C_\phi T_g$ is compact, we actually deduce that $\|(C_\phi T_g)f_n\|_{H^p} \rightarrow 0$. Hence (3.10) yields that $\|(\chi_E T_g)f_n\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$, and this finishes the proof of Proposition 3.2. $\square$

Note that altogether the above steps complete the proof of Theorem 3.1.

Remark 3.5. (1) Theorem 1.1 and Corollary 1.2 state that the linear qualitative behaviour of non-compact Volterra operators $T_g: H^p \rightarrow H^p$ for $p \neq 2$ is very restricted compared to that of arbitrary bounded operators on H^p . Recall that a result of Weis [23] for $L^p(0, 1)$ combined with the known isomorphism $H^p \approx L^p(\mathbb{T}, m)$ for $1 < p < \infty$, see [5], imply that if the restriction $U|_M$ of a given operator U on H^p is bounded below on some infinite-dimensional subspace $M \subset H^p$, then M contains an isomorphic copy of either ℓ^p or ℓ^2 . We also remind that if $p > 2$ then any infinite-dimensional subspace $M \subset H^p$ contains isomorphic copies of either ℓ^p or ℓ^2 , but the case $1 < p < 2$ is much more complicated, see e.g. [1, Chap. 6.4] for results of this type. The case of the Hilbert space H^2 is well known and different, since $\mathcal{S}_2(H^2)$ (i.e. the compact operators) is the unique closed ideal in the algebra of bounded operators, cf. [18, 5.1–5.2].

(2) The crucial estimate (3.10) on a compact set E having large measure, which is used in the proof of Proposition 3.2, can also be obtained in a direct fashion without recourse to the result by F. and M. Riesz. In fact, let $\varepsilon > 0$ be given (the exact value will be specified later) and write $F = \{\zeta \in K : h(\zeta) > 1/2\}$. Since $h \leq 1$ a.e., one has

$$\nu(K) = \int_K h dm \leq m(\mathbb{T} \setminus F) \cdot \frac{1}{2} + m(F) = \frac{1}{2}(1 + m(F)),$$

and hence

$$(3.11) \quad m(F) \geq 2\nu(K) - 1 = 1 - 2\nu(\partial\Omega \setminus K).$$

Write $\mathbb{T} \setminus K = \bigcup_j I_j$, where the open arcs $I_j \subset \mathbb{T}$ are the (disjoint) connected components of $\mathbb{T} \setminus K$. For each j , let ζ_j be the midpoint of I_j and consider the Carleson window $W_j = W(\zeta_j, 2\pi m(I_j))$ (if $m(I_j) \geq 1/2\pi$, take $W_j = \mathbb{D}$). It follows from the geometry of Ω that $\partial\Omega \subset K \cup (\bigcup_j \overline{W_j})$. Since ν is a Carleson measure, we have $\nu(\overline{W_j}) \leq cm(I_j)$ where $c > 0$ is an absolute constant (recall that $\phi(0) = 0$). Consequently,

$$\nu(\partial\Omega \setminus K) \leq c \sum_j m(I_j) = cm(\mathbb{T} \setminus K) < c\varepsilon.$$

In conjunction with (3.11), this gives $m(F) \geq 1 - 2c\varepsilon$. On choosing $\varepsilon = \eta/4c$ and a compact set $E \subset F$ with $m(F \setminus E) < \eta/2$, we get (3.10) with $\delta = 1/2$ and $m(\mathbb{T} \setminus E) < \eta/2 + \eta/2 = \eta$.

(3) Theorem 1.1 also holds, with a similar proof to the case $p = 1$, in the quasi-normed range $0 < p < 1$. However, here much less is known about the linear qualitative classification of operators, so we will not include the technical details of this extension.

In the proof of Lemma 3.4 we utilized the compactness of certain Volterra composition operators $C_\phi T_g$ on H^p for arbitrary $g \in BMOA \setminus VMOA$ and an associated conformal map ϕ . General operators of this kind were first considered by Li and Stević [15], and they have subsequently been studied in several papers. As a simple by-product of part of the proof of Lemma 3.4 we also record a characterisation of the compactness of arbitrary products $C_\phi T_g: H^p \rightarrow H^p$, which to the best of our knowledge has not been made explicit in the literature.

We first point out a useful formula for the norm $\|(C_\phi T_g)f\|_{H^2}$. Let N_ϕ be the Nevanlinna counting function of ϕ , that is, $N_\phi(w) = \sum_{z \in \phi^{-1}(w)} \log(1/|z|)$ for $w \in \phi(\mathbb{D})$ and $w \neq \phi(0)$, and $N_\phi(w) = 0$ for $w \notin \phi(\mathbb{D})$.

Lemma 3.6. *Suppose that $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic map, $g \in BMOA$ and $f \in H^2$. Then*

$$\|(C_\phi T_g)f\|_{H^2}^2 = |T_g f(\phi(0))|^2 + 2 \int_{\mathbb{D}} |f(w)|^2 |g'(w)|^2 N_\phi(w) dA(w).$$

Proof. By applying a norm formula for composition operators due to J. Shapiro, see e.g. [7, Thm 2.31], which combines the Littlewood-Paley identity with a change of variables (a special case of Stanton's formula), it follows that

$$\begin{aligned} \|(C_\phi T_g)f\|_{H^2}^2 &= |T_g f(\phi(0))|^2 + 2 \int_{\mathbb{D}} |(T_g f)'(w)|^2 N_\phi(w) dA(w) \\ &= |T_g f(\phi(0))|^2 + 2 \int_{\mathbb{D}} |f(w)|^2 |g'(w)|^2 N_\phi(w) dA(w). \end{aligned}$$

□

Corollary 3.7. *Suppose that $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic map, $g \in BMOA$ and $0 < p < \infty$. Then $C_\phi T_g$ is compact $H^p \rightarrow H^p$ if and only if $|g'|^2 N_\phi dA$ is a vanishing Carleson measure.*

Proof. Suppose first that $p = 2$ and let $(f_n) \subset H^2$ be any weak-null sequence, that is, (f_n) is bounded and $f_n \rightarrow 0$ uniformly on the compact subsets of $\mathbb{D}$. It follows that

$$T_g f_n(\phi(0)) = \int_0^{\phi(0)} f_n(w) g'(w) dw \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, if $|g'|^2 N_\phi dA$ is a vanishing Carleson measure, then the embedding $H^2 \rightarrow L^2(\mathbb{D}, |g'|^2 N_\phi dA)$ is compact, and Lemma 3.6 implies that $\|(C_\phi T_g)f_n\|_{H^2} \rightarrow 0$ as $n \rightarrow \infty$. This entails that $C_\phi T_g: H^2 \rightarrow H^2$ is compact. Moreover, since $C_\phi T_g$ is bounded $H^p \rightarrow H^p$ for any $0 < p < \infty$, the compactness of $C_\phi T_g: H^p \rightarrow H^p$ can be deduced from one-sided Krasnoselskii interpolation as in the proof of Lemma 3.4.

Conversely, if $C_\phi T_g: H^p \rightarrow H^p$ is compact for some $p \in (0, \infty)$, then interpolation yields compactness for $p = 2$, and again Lemma 3.6 yields that the embedding $H^2 \rightarrow L^2(\mathbb{D}, |g'|^2 N_\phi dA)$ is compact. This means that $|g'|^2 N_\phi dA$ is a vanishing Carleson measure, see e.g. the proof of [7, Thm 2.33]. □

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