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The H-infinity-functional calculus

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The H^∞ -functional calculus

Now that we have a variety of probabilistic and operator-theoretic tools at our disposal it is time to start using them. The present chapter is devoted to the study of sectorial operators (this class of operators includes, e.g., uniformly elliptic second order differential operators) and their holomorphic functional calculus formally defined by the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma} f(z)(z - A)^{-1} dz.$$

Here f is a bounded holomorphic function on a sector Σ about the positive real line containing the spectrum of A , and on the complement of $\overline{\Sigma}$ the parabolic bound

$$\|(z - A)^{-1}\| \lesssim \frac{1}{|z|}$$

is assumed. Unless f has additional integrability properties near the origin and infinity, this Dunford integral is a singular integral and the operator $f(A)$ can be defined only as an unbounded operator. If $f(A)$ is bounded and satisfies

$$\|f(A)\| \lesssim \|f\|_\infty$$

for all bounded holomorphic function on Σ , then A is said to have a bounded H^∞ -calculus on Σ . For classical operators such as $A = -\Delta$ on $L^p(\mathbb{R}^d)$, the boundedness of the H^∞ -calculus can be proved using Fourier analysis methods such as the Fourier multiplier theorems of Volume I and classical square function estimates. With this model in mind we use the probabilistic and operator-theoretic techniques of earlier chapters of the present volume to develop the theory of H^∞ -calculus for general sectorial operators acting on a Banach. In particular we study their R -boundedness properties and characterisations in terms of the generalised square functions developed in the previous chapter. Both will play a crucial role in Volume III as a bridge connecting estimates from harmonic analysis, stochastic analysis, and operator theory. This

will be particularly useful in the theory of maximal regularity for solutions of evolution equations of parabolic type.

To give an impression of how the Banach space properties studied in earlier chapters enter the picture we mention the facts, proved in this chapter, that the Laplace operator on $L^p(\mathbb{R}^d; X)$, $1 < p < \infty$, has a bounded H^∞ -calculus if and only if X is a UMD space, and that in this situation the range of its H^∞ -calculus is γ -bounded if and only if X also has Pisier's contraction property.

Besides studying the H^∞ -calculus for its own sake, we provide three important classes of examples of operators admitting a bounded H^∞ -calculus:

- If A is the generator of a C_0 -semigroup of contractions on a Hilbert space, then A has a bounded H^∞ -calculus;
- If A is the generator of a C_0 -semigroup of positive contractions on an L^p -space, then A has a bounded H^∞ -calculus.
- If iA is the generator of a bounded C_0 -group on a UMD space, then A has a bounded H^∞ -calculus.

After a careful study of the H^∞ -calculus for sectorial operators, in the second part of the chapter we also consider bisectorial operators and, in particular, generators of bounded C_0 -groups. Their functional calculus properties are intimately connected to those of sectorial operators, a fact which we exploit to pass from third example in our list, which can be treated by means of transference techniques, to the second via dilation arguments. In the concluding section we combine the ℓ^p -boundedness techniques of Chapter 8 with interpolation theory and an ergodic theorem to give a substantial improvement of the second example in case the semigroup has a bounded analytic extension to a sector.

10.1 Sectorial operators

For $0 < \omega < \pi$ we denote by

$$\Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega\}$$

the open sector of angle ω in the complex plane; the argument is taken in $(-\pi, \pi)$. For $z \in \varrho(A)$, the resolvent set of A , let $R(z, A) := (z - A)^{-1}$ denote the resolvent of A . Any operator with non-empty resolvent set is closed.

Definition 10.1.1 (Sectorial operators). *A linear operator $(A, \mathcal{D}(A))$ is said to be sectorial if there exists $\omega \in (0, \pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$ and*

$$M_{\omega, A} := \sup_{z \in \mathbb{C} \setminus \overline{\Sigma_\omega}} \|zR(z, A)\| < \infty.$$

In this situation we say that A is ω -sectorial with constant $M_{\omega,A}$. The infimum of all $\omega \in (0, \pi)$ such that A is ω -sectorial is called the angle of sectoriality of A and is denoted by $\omega(A)$.

Fig. 10.1: The spectrum of a sectorial operator

The reader is cautioned that the definition of sectoriality varies in the literature. As we will see, various results admit a somewhat cleaner statement if the requirement that A have dense range is added to the definition, and this is indeed what some authors do. This has the further benefit that sectorial operators with dense range are automatically injective. What is more, imposing dense range is hardly a restriction, for the following reason. If the Banach space X is reflexive (and more generally, if the resolvent operators of A are weakly compact), then every sectorial operator A on X is densely defined and we have a direct sum decomposition

$$X = \mathbf{N}(A) \oplus \overline{\mathbf{R}(A)}. \quad (10.1)$$

Moreover, the part of A in $\overline{\mathbf{R}(A)}$ is sectorial, densely defined, has dense range, and is injective, while the part of A in $\mathbf{N}(A)$ equals the zero operator. Thus all useful information about A is contained in the part of A in $\overline{\mathbf{R}(A)}$. The reader is referred to Propositions 10.1.8 and 10.1.9 for the full statement of these results.

There are various reasons for insisting on the more general definition of a sectorial operator adopted here. Firstly, some natural sectorial operators fail to have dense range. These include the Laplace operator on $L^1(\mathbb{R}^d)$, the Neumann Laplacian on $L^p(D)$ with $D \subseteq \mathbb{R}^d$ bounded, and the Ornstein-Uhlenbeck operator on $L^p(\mathbb{R}^d, \gamma_d)$, with γ_d the standard Gaussian measure on

$\mathbb{R}^d$. Secondly, with our definition of sectoriality, the adjoint A^* of a densely defined sectorial operator is sectorial, but with an additional dense range condition this statement would no longer be true, as in general the range of A^* is weak*-dense only. Thirdly, and perhaps most importantly, in our approach it becomes transparent which results need additional assumptions and which do not. In general, the need of additional assumptions is reflected in limitations on range of applicability. For instance, in the treatment of the H^∞ -calculus associated with a sectorial operator A , as a rule of thumb statements about $f(A)x$ for functions in $f \in H^1(\Sigma)$ tend to be true for general elements x in X , whereas statements involving functions $f \in H^\infty(\Sigma)$ usually require the consideration of elements x in $D(A) \cap R(A)$ or its closure.

10.1.a Examples

Before turning to some basic properties of sectorial operators it is instructive to present some examples. With an eye towards the above discussion, in each of these examples we pay attention as to whether or not the operator has dense range and/or is injective. We will freely use the terminology and results of Appendix G on semigroups of operators.

The first two examples describe the relationship between sectorial operators and bounded (analytic) semigroups.

Example 10.1.2 (Sectorial operators and bounded semigroups). If a linear operator $-A$ generates a uniformly bounded C_0 -semigroup $(S(t))_{t \geq 0}$ on a Banach space X , then A is densely defined, closed, and ω -sectorial for all $\frac{1}{2}\pi < \omega < \pi$. In particular, $\omega(A) = \frac{1}{2}\pi$. Indeed, by Proposition G.4.1, the open left half-plane $\mathbb{C}_- = \{\Re \lambda < 0\}$ is contained in the resolvent set of A and we have the bound

$$\|R(\lambda, A)\| \leq \frac{M}{|\Re \lambda|}$$

for all $\lambda \in \mathbb{C}_-$. Moreover, if $|\arg(\lambda)| \geq \frac{1}{2}\pi + \nu$, with $\nu \in (0, \frac{1}{2}\pi)$, then

$$\|R(\lambda, A)\| \leq \frac{M}{|\Re \lambda|} \leq \frac{M}{\sin \nu} \frac{1}{|\lambda|}$$

and A is $\frac{1}{2}\pi + \nu$ -sectorial.

Example 10.1.3 (Sectorial operators and bounded analytic semigroups). Theorem G.5.2 may be formulated as stating that for a densely defined operator A on a Banach space X ,

$$A \text{ is sectorial of angle } 0 < \theta < \frac{1}{2}\pi \quad (10.2)$$

if and only if

$$-A \text{ generates a bounded analytic } C_0\text{-semigroup } (S(t))_{t \geq 0} \quad (10.3)$$

on Σ_η for some $0 < \theta < \frac{1}{2}\pi$.

Moreover,

$$\inf \{ \theta \in (0, \pi/2) : (10.2) \text{ holds} \} = \pi/2 - \sup \{ \eta \in (0, \pi/2) : (10.3) \text{ holds} \}.$$

We continue with some concrete examples, restricting ourselves to some model cases which can be treated here in a self-contained manner. More sophisticated examples will be discussed in the Notes at the end of the chapter. The examples presented here will be revisited several times as the chapter unfolds, since they serve well to illustrate the various concepts we are about to discuss.

Example 10.1.4 (First derivative). The first derivative $Af = f'$ with domain $D(A) = W^{1,p}(\mathbb{R}; X)$ is ω -sectorial on $L^p(\mathbb{R}; X)$, $1 \leq p < \infty$, for all $\frac{1}{2}\pi < \omega < \pi$. In order to prove this it suffices to recall that in Example G.4.4 we have checked that $-A$ generates the translation group, and therefore Example 10.1.2 implies that A is ω -sectorial for all $\frac{1}{2}\pi < \omega < \pi$.

Lemma 2.5.11 implies that A is injective. To show that $R(A)$ is dense for $p \in (1, \infty)$, first consider the scalar-valued case. The direct sum decomposition (10.1) implies that the range of A is dense in $L^p(\mathbb{R})$. Denoting the range in the scalar-valued case by R , the range of A in the vector-valued case contains $R \otimes X$, which is dense in $L^p(\mathbb{R}; X)$.

The operator A does not have dense range for $p = 1$. Indeed, approximating $f \in D(A)$ in the norm of $W^{1,1}(\mathbb{R}; X) = D(A)$ by test functions $f_n \in C_c^1(\mathbb{R}^d; X)$, one sees that

$$\langle Af, \mathbf{1} \otimes x^* \rangle = \lim_{n \rightarrow \infty} \langle f'_n, \mathbf{1} \otimes x^* \rangle = \lim_{r \rightarrow \infty} \int_{-r}^r \frac{d}{dt} \langle f_n(t), x^* \rangle dt = 0.$$

Since $\langle Af, \mathbf{1} \otimes x^* \rangle = \int_{\mathbb{R}} \langle Af(t), x^* \rangle dt = \langle \int_{\mathbb{R}} Af(t) dt, x^* \rangle$, it follows that $\int_{\mathbb{R}} Af(t) dt = 0$. Thus we see that $R(A)$ is contained in the proper closed subspace $\{g \in L^1(\mathbb{R}; X) : \int_{-\infty}^{\infty} g(t) dt = 0\}$ of $L^1(\mathbb{R}; X)$.

This example is somewhat deceptive in that it conceals a much better property of the first derivative, namely that the operators $\pm iA = \pm id/dx$ are *bisectorial* of angle 0. The notion of bisectoriality will be discussed at length in Section 10.6.

Example 10.1.5 (Laplacian on $\mathbb{R}^d$). In Example G.5.6 it is shown that the Laplacian Δ on $L^p(\mathbb{R}^d; X)$, $1 \leq p < \infty$, with domain $D(\Delta) = H^{2,p}(\mathbb{R}; X)$, generates an analytic C_0 -semigroup which is bounded on every sector Σ_η , $\eta \in (0, \frac{1}{2}\pi)$. Thus Example 10.1.3 implies that $-\Delta$ is sectorial of zero angle.

Let us show that Δ is injective. If $f \in D(\Delta)$ satisfies $\Delta f = 0$, then for all $x^* \in X^*$ the function $\langle f, x^* \rangle \in L^p(\mathbb{R}^d)$ belongs to the domain of the scalar-valued Laplacian and $\langle \Delta f, x^* \rangle = \Delta \langle f, x^* \rangle$. Thus it suffices to prove injectivity in the scalar case; for once we have that, in the vector-valued case it will imply that $\langle f, x^* \rangle = 0$ for all $x^* \in X^*$, and therefore $f = 0$ by Corollary 1.1.25.

Suppose then that the scalar-valued function $f \in D(\Delta)$ satisfies $\Delta f = 0$. Replacing f by $f * \phi$ for some test function (and noting that $f = 0$ if $f * \phi = 0$ for all test functions ϕ), we may assume that f is smooth. Then, by the maximum principle, $\Delta f = 0$ implies that f is constant. Since also $f \in L^p(\mathbb{R}^d)$, this is only possible if $f = 0$.

For $1 < p < \infty$ the density of the range of Δ can be proved as in Example 10.1.4. For $p = 1$, $\langle \Delta f, \mathbf{1} \otimes x^* \rangle = 0$ implies that $R(\Delta)$ is contained in the proper closed subspace $\{f \in L^1(\mathbb{R}^d; X) : \int_{\mathbb{R}^d} f(x) dx = 0\}$ of $L^1(\mathbb{R}^d; X)$.

10.1.b Basic properties

We now turn to the study of some basic properties of sectorial operators. Readers familiar with the topic may safely skip this discussion and proceed immediately to Section 10.2.a.

We begin with the simple observation that sectoriality can be checked on a half-line (see Lemma G.1.4):

Proposition 10.1.6. *A linear operator A is sectorial if and only if the open half-line $(-\infty, 0)$ is contained in $\varrho(A)$ and $\sup_{\lambda < 0} \|\lambda R(\lambda, A)\| < \infty$. More precisely, if*

$$\sup_{\lambda < 0} \|\lambda R(\lambda, A)\| =: M < \infty,$$

then A is sectorial with $\omega(A) \leq \pi - \arcsin(1/M)$.

Proposition 10.1.7 implies that $M \geq 1$, so that $\arcsin(1/M)$ is well defined. The next proposition gives useful characterisations of the kernel and the closure of the range of a sectorial operator.

Proposition 10.1.7. *Let A be sectorial.*

(1) *For all $x \in \overline{D(A)}$,*

$$\lim_{t \rightarrow \infty} t(t + A)^{-1}x = x, \quad \lim_{t \rightarrow \infty} A(t + A)^{-1}x = 0. \quad (10.4)$$

(2) *For all $x \in \overline{R(A)}$,*

$$\lim_{t \downarrow 0} t(t + A)^{-1}x = 0, \quad \lim_{t \downarrow 0} A(t + A)^{-1}x = x. \quad (10.5)$$

(3) *$N(A) \cap \overline{R(A)} = \{0\}$ and $N(A) \oplus \overline{R(A)} = \{x \in X : \lim_{t \downarrow 0} t(t + A)^{-1}x \text{ exists}\}$.*

Proof. (1): If $x \in D(A)$ then

$$\lim_{t \rightarrow \infty} t(t + A)^{-1}x - x = - \lim_{t \rightarrow \infty} A(t + A)^{-1}x = - \lim_{t \rightarrow \infty} \frac{1}{t} [t(t + A)^{-1}]Ax = 0$$

by the uniform boundedness of the operators $t(t + A)^{-1}$, $t > 0$. By an approximation argument using the uniform boundedness of the operators $t(t + A)^{-1}$ and $A(t + A)^{-1}$, the conclusion extends to all $x \in \overline{D(A)}$.

(2): If $x \in \mathbf{R}(A)$, say $x = Ay$ with $y \in \mathbf{D}(A)$, then

$$\lim_{t \downarrow 0} t(t + A)^{-1}x = -\lim_{t \downarrow 0} A(t + A)^{-1}x + x = \lim_{t \downarrow 0} t(1 - t(t + A)^{-1})y = 0.$$

As in the proof of (1), by an approximation argument the conclusion extends to all $x \in \overline{\mathbf{R}(A)}$.

(3): For all $x \in \overline{\mathbf{R}(A)}$ by (2) we have $\lim_{t \downarrow 0} t(t + A)^{-1}x = 0$. On the other hand, for all $x \in \mathbf{N}(A)$ we have

$$\lim_{t \downarrow 0} t(t + A)^{-1}x = x - \lim_{t \downarrow 0} (t + A)^{-1}Ax = x.$$

It follows that $\mathbf{N}(A) \cap \overline{\mathbf{R}(A)} = \{0\}$.

Turning to the second assertion, by the above the asserted limits exist for all $x \in \mathbf{N}(A) \oplus \overline{\mathbf{R}(A)}$. Conversely, suppose $x \in X$ is such that $\lim_{t \downarrow 0} t(t + A)^{-1}x =: y$ exists. Then

$$A(t + A)^{-1}x = x - t(t + A)^{-1}x \rightarrow x - y$$

as $t \downarrow 0$. It follows that $x - y \in \overline{\mathbf{R}(A)}$. It also follows that $tA(t + A)^{-1}x \rightarrow 0$ as $t \downarrow 0$. Since the graph of A is closed, this implies that $y \in \mathbf{D}(A)$ and $Ay = 0$. Hence $x = y + (x - y)$ with $x - y \in \overline{\mathbf{R}(A)}$ and $y \in \mathbf{N}(A)$. $\square$

If Y is a subspace of X and A is linear operator on X with domain $\mathbf{D}(A)$, the part of A in Y is the linear operator A_Y on Y with domain $\mathbf{D}(A_Y)$ defined by

$$\begin{aligned} \mathbf{D}(A_Y) &:= \{y \in \mathbf{D}(A) \cap Y : Ay \in Y\}, \\ A_Y y &:= Ay, \quad y \in \mathbf{D}(A_Y). \end{aligned}$$

The intersection $\mathbf{D}(A) \cap \mathbf{R}(A)$ plays a central role in the whole subject. The next proposition shows, among other things, that the part of A in the closure of this subspace has particularly good properties.

Proposition 10.1.8. *Let A be ω -sectorial for some $0 < \omega < \pi$.*

- (1) $\mathbf{D}(A) \cap \mathbf{R}(A) = \mathbf{R}(A(I + A)^{-2})$.
- (2) $\overline{\mathbf{D}(A) \cap \mathbf{R}(A)} = \overline{\mathbf{D}(A)} \cap \overline{\mathbf{R}(A)} = \mathbf{R}(A(I + A)^{-2})$.
- (3) *The part of A in $\overline{\mathbf{D}(A) \cap \mathbf{R}(A)}$ is closed, densely defined, injective, has dense range, and is ω -sectorial.*
- (4) *If A is densely defined, then $\overline{\mathbf{D}(A) \cap \mathbf{R}(A)} = \overline{\mathbf{R}(A)}$.*

Proof. (1): For $x \in X$, the identity

$$A(I + A)^{-2}x = (I + A)^{-1}A(I + A)^{-1}x$$

shows “ $\supseteq$ ”. If $x \in \mathbf{D}(A) \cap \mathbf{R}(A)$, then $(I + A)x \in \mathbf{R}(A)$, say $(I + A)x = Ay$ for some $y \in \mathbf{D}(A)$. Let $z := (I + A)y$. Then

$$x = (I + A)^{-1}Ay = A(I + A)^{-2}z.$$

(2): It suffices to prove the inclusion $\overline{D(A)} \cap \overline{R(A)} \subseteq \overline{D(A) \cap R(A)}$. Let $x \in \overline{D(A)} \cap \overline{R(A)}$ and fix $\varepsilon > 0$. By (10.4) we can find $t > 0$ such that

$$\|x - t(t + A)^{-1}x\| < \frac{\varepsilon}{2}.$$

Since $y = t(t + A)^{-1}x$ belongs to $\overline{R(A)}$, by (10.5) we can find $s > 0$ such that

$$\|y - A(s + A)^{-1}y\| < \frac{\varepsilon}{2}.$$

It follows that

$$\|x - tA(s + A)^{-1}(t + A)^{-1}x\| < \varepsilon$$

and the result follows since $tA(s + A)^{-1}(t + A)^{-1}x \in D(A) \cap R(A)$.

(3): Let us write $Y = \overline{D(A) \cap R(A)}$ for brevity and denote by A_Y the part of A in Y . Since A is closed it is clear that A_Y is closed. Since $(\lambda - A)^{-1}$ maps $D(A) \cap R(A)$ into itself, it also maps Y into itself, and from this it follows that if $\lambda \in \varrho(A)$ then $\lambda \in \varrho(A_Y)$ and $(\lambda - A_Y)^{-1} = (\lambda - A)^{-1}|_Y$. As a result, A_Y is ω -sectorial on Y .

We prove next that A_Y has dense range in Y and is injective. If $y \in D(A) \cap R(A)$, then by (10.5),

$$y = \lim_{t \downarrow 0} A(t + A)^{-1}y = \lim_{t \downarrow 0} A_Y(t + A)^{-1}y$$

belongs to $\overline{R(A_Y)}$. This being true for all y in the dense subspace $D(A) \cap R(A)$ of Y , it follows that $R(A_Y) = Y$. By the result of Proposition 10.1.7(2) it follows that $N(A_Y) = \{0\}$.

We show next that $D(A_Y)$ is dense in Y . If $y \in D(A) \cap R(A)$, then $\lim_{t \downarrow 0} A(t + A)^{-1}y = y$ by (10.5). Moreover, for each $t > 0$,

$$\lim_{n \rightarrow \infty} n(n + A)^{-1}A(t + A)^{-1}y = A(t + A)^{-1}y;$$

here we used that $A(t + A)^{-1}y = (t + A)^{-1}Ay$ belongs to $D(A)$ together with (10.4). Since the vectors $n(n + A)^{-1}A(t + A)^{-1}y$ belong to $D(A_Y)$ (because

$$nA(n + A)^{-1}A(t + A)^{-1}y = nA(t + A)^{-1}y - n^2A(n + A)^{-1}(t + A)^{-1}y$$

belongs to Y), it follows that $A(t + A)^{-1}y \in \overline{D(A_Y)}$ for all $t > 0$, and therefore also $y \in \overline{D(A_Y)}$. This proves that $D(A) \cap R(A) \subseteq \overline{D(A_Y)}$ and therefore $\overline{D(A_Y)} = Y$.

(4): If A is densely defined, then by (2) we have $\overline{D(A) \cap R(A)} = \overline{D(A)} \cap \overline{R(A)} = \overline{R(A)}$. $\square$

The final result of this subsection shows that in reflexive Banach spaces X , sectorial operators are automatically densely defined, so it always holds that $\overline{D(A) \cap R(A)} = \overline{R(A)}$. Even more importantly, X decomposes as the direct sum of $N(A)$ and $\overline{R(A)}$:

Proposition 10.1.9. *Let A be an ω -sectorial operator on a reflexive Banach space X for some $0 < \omega < \pi$. Then A is densely defined and therefore*

$$\overline{D(A) \cap R(A)} = \overline{R(A)}.$$

Furthermore, we have a direct sum decomposition

$$X = N(A) \oplus \overline{R(A)}.$$

Proof. We begin with the proof of the direct sum decomposition. Since X is reflexive the resolvent operators $(t_0 + A)^{-1}$ are weakly compact for all $t_0 > 0$. By the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \quad \lambda, \mu \in \varrho(A),$$

we have

$$(t + A)^{-1} = (t_0 + A)^{-1}[I + (t_0 - t)(t + A)^{-1}],$$

which implies that for each $x \in X$, $\{t(t + A)^{-1}x : t \in (0, 1)\}$ is contained in some scalar multiple of the relatively compact set $(t_0 + A)^{-1}B_X$, with B_X the unit ball of X . As a consequence, for every $x \in X$ we may extract a subsequence $t_n \downarrow 0$ such that the weak limit $\lim_{n \rightarrow \infty} t_n(t_n + A)^{-1}x =: y$ exists. Using that closed subspaces are weakly closed, we may now argue as in the proof of Proposition 10.1.7(3) to conclude that $x = y + (x - y)$ with $y \in N(A)$ and $x - y \in \overline{R(A)}$.

To prove that A is densely defined we will show that the part A_Y is densely defined in $Y := \overline{R(A)}$. Since $N(A)$ is contained in $D(A)$ the decomposition then implies that A is densely defined in X .

Fix an arbitrary $y \in Y$. Since the operators $n(n + A_Y)^{-1}$ are uniformly bounded, we may select a subsequence $n_k \rightarrow \infty$ such that $n_k(n_k + A_Y)^{-1}y$ is weakly convergent, say to $x \in X$. Then $x \in Y$ and, by the resolvent identity,

$$\begin{aligned} (1 + A_Y)^{-1}y &= \lim_{k \rightarrow \infty} n_k(n_k + A_Y)^{-1}(1 + A_Y)^{-1}x \\ &= \lim_{k \rightarrow \infty} \frac{n_k(n_k + A_Y)^{-1}x + n_k(1 + A_Y)^{-1}x}{\mu + n_k} = (1 + A_Y)^{-1}x, \end{aligned}$$

where the limits are taken in the weak topology. It follows that $y = x = \lim_{k \rightarrow \infty} n_k R(-n_k, A_Y)y$ weakly, which shows that y belongs to the weak closure of $D(A_Y)$. Since this equals the strong closure by the Hahn–Banach theorem, this completes the proof. $\square$

Remark 10.1.10. The proof of the direct sum decomposition extends to arbitrary Banach spaces as long as A has weakly compact resolvent operators (the resolvent identity implies that if $R(\lambda, A)$ is weakly compact for some $\lambda \in \varrho(A)$, then it is weakly compact for all $\lambda \in \varrho(A)$).

As is shown in the following example, a general sectorial operator A need not be densely defined and the direct sum decomposition of Proposition 10.1.9 may fail.

Example 10.1.11. Consider the bounded linear operator A on $X = C[0, 1]$ by $Af(s) = sf(s)$. This operator is injective and sectorial of angle zero, and $N(A) \oplus \overline{R(A)} = \overline{R(A)} = \{f \in C[0, 1] : f(0) = 0\}$ is a proper subspace of $C[0, 1]$.

The inverse operator A^{-1} with $D(A^{-1}) = \{f \in C[0, 1] : f(0) = 0\}$ provides an example of a sectorial operator of angle zero which does not have a dense domain.

10.2 Construction of the H^∞ -calculus

The construction of the H^∞ -calculus proceeds in two stages. The first generalises the Dunford functional calculus

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, A) \, dz$$

for bounded operators A and f a holomorphic function in a neighbourhood of its spectrum $\sigma(A)$, to sectorial operators A . The requirement will be that f belongs to the Hardy space $H^1(\Sigma_\sigma)$, where $\omega(A) < \sigma < \pi$. This is by itself not a very useful calculus, and the second stage of the construction consists of extending the Dunford calculus to suitable *bounded* holomorphic functions on Σ , that is, for functions in the Hardy space $H^\infty(\Sigma)$. Whether or not this is possible depends on the operator A , and A is said to have a *bounded H^∞ -functional calculus* if there exists a constant M such that

$$\|f(A)\| \leq M\|f\|_\infty, \quad f \in H^1(\Sigma) \cap H^\infty(\Sigma). \quad (10.6)$$

When (10.6) holds, the restriction of the H^1 -calculus to functions in $H^1(\Sigma) \cap H^\infty(\Sigma)$ can be extended to a calculus for functions in $H^\infty(\Sigma)$ by an approximation argument.

The preliminary H^1 -calculus, which we will call the *Dunford calculus* in view of its close analogy with the Dunford calculus for bounded linear operators, is “easy” in the sense that it is defined, for any sectorial operator, in terms of absolutely convergent integrals. In contrast, the H^∞ -calculus involves singular integrals which do not always converge. For operators for which they do converge, the H^∞ -calculus provides a powerful tool with a wealth of non-trivial applications.

10.2.a The Dunford calculus

We begin by setting up a preliminary calculus $f \mapsto f(A)$ for sectorial operators A and functions f which are holomorphic and integrable on an open sector whose opening angle is somewhat larger than the angle of sectoriality of A . This calculus, the so-called *Dunford calculus* of A , involves an absolutely convergent integral of Dunford type.

Definition 10.2.1 (The Hardy spaces $H^p(\Sigma_\sigma)$). Let $1 \leq p \leq \infty$ and $0 < \sigma < \pi$. The Banach space of all holomorphic functions $f : \Sigma_\sigma \rightarrow \mathbb{C}$ for which

$$\|f\|_{H^p(\Sigma_\sigma)} := \sup_{|\nu| < \sigma} \|t \mapsto f(e^{i\nu}t)\|_{L^p(\mathbb{R}_+, \frac{dt}{t})} < \infty$$

is denoted by $H^p(\Sigma_\sigma)$.

Appendix H contains some generalities about holomorphic functions on sectors which will be used freely in this chapter. Among other things, it is shown there that by restriction, every function $f \in H^p(\Sigma_\sigma)$ with $1 \leq p \leq \infty$ belongs to $H^\infty(\Sigma_{\sigma'})$ for all $0 < \sigma' < \sigma$.

Suppose now that A is sectorial, with angle of sectoriality $\omega(A)$. Recall that this means that for all $\omega(A) < \sigma < \pi$ we have $\mathbb{C}\overline{\Sigma_\sigma} \subseteq \varrho(A)$ and

$$M_{\sigma,A} := \sup_{z \in \mathbb{C}\overline{\Sigma_\sigma}} \|zR(z, A)\| < \infty.$$

We will refer to $M_{\sigma,A}$ as the σ -sectoriality constant of A . This constant depends continuously on $\sigma \in (\omega(A), \pi)$.

For functions $f \in H^1(\Sigma_\sigma)$ we may define a bounded operator $f(A)$ by the Dunford integral

$$f(A) := \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z)R(z, A) dz, \quad (10.7)$$

where $\omega(A) < \nu < \sigma$ is chosen arbitrarily; the boundary $\partial\Sigma_\nu$ is oriented ‘downwards’ so as to have the spectrum of A at the left-hand side while traversing it. The estimate

$$\|f(A)\| \leq \frac{M_{\nu,A}}{2\pi} \int_{\partial\Sigma_\nu} |f(z)| \frac{|dz|}{|z|} \leq \frac{M_{\nu,A}}{\pi} \|f\|_{H^1(\Sigma_\sigma)}$$

shows that the integral in (10.7) is well defined as a Bochner integral in $\mathcal{L}(X)$. Taking the infimum over all $\nu \in (\omega(A), \sigma)$ gives the bound

$$\|f(A)\| \leq \frac{M_{\sigma,A}}{\pi} \|f\|_{H^1(\Sigma_\sigma)}. \quad (10.8)$$

Next we show that the definition of $f(A)$ is independent of the choice of $\nu \in (\omega(A), \sigma)$. Letting $\text{Arc}(r, \eta) = \{re^{iy} \in \mathbb{C} : |y| < \eta\}$ for $r > 0$ and $\eta \in (0, \sigma)$, we have

$$\int_{\text{Arc}(r, \eta)} |f(z)| \frac{|dz|}{|z|} = \int_{-\eta}^{\eta} |f(re^{iy})| |y| dy \rightarrow 0,$$

as $r \rightarrow \infty$ or $r \downarrow 0$ by Proposition H.2.5. Therefore, since $z \mapsto f(z)R(z, A)$ is holomorphic on $\Sigma_\sigma \setminus \overline{\Sigma_{\omega(A)}}$, one can use Cauchy's theorem to see that the right-hand side of (10.7) does not depend on the choice of $\nu \in (\omega(A), \sigma)$.

By a change of variables, for all $t > 0$ we have $\|z \mapsto f(tz)\|_{H^1(\Sigma_\sigma)} = \|f\|_{H^1(\Sigma_\sigma)}$ and consequently (10.8) self-improves to

$$\sup_{t>0} \|f(tA)\| \leq \frac{M_{\sigma, A}}{\pi} \|f\|_{H^1(\Sigma_\sigma)}. \quad (10.9)$$

Theorem 10.2.2 (Dunford calculus). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. The mapping $f \mapsto f(A)$ from $H^1(\Sigma_\sigma)$ to $\mathcal{L}(X)$ is linear, and multiplicative in the sense that if $f, g \in H^1(\Sigma_\sigma)$ are such that $fg \in H^1(\Sigma_\sigma)$, then*

$$(fg)(A) = f(A)g(A).$$

It has the following convergence property: if the functions $f_n, f \in H^\infty(\Sigma_\sigma)$ are uniformly bounded and satisfy $f_n(z) \rightarrow f(z)$ for all $z \in \Sigma_\sigma$, then for all $g \in H^1(\Sigma_\sigma)$ we have

$$\lim_{n \rightarrow \infty} (f_n g)(A) = (fg)(A). \quad (10.10)$$

Proof. Linearity of the mapping $f \mapsto f(A)$ is clear from the definitions. To prove multiplicativity we choose angles $\omega(A) < \nu' < \nu < \sigma' < \sigma$. Then, for $f \in H^1(\Sigma_\sigma)$ and $g \in H^1(\Sigma_{\sigma'})$ such that $fg \in H^1(\Sigma_\sigma)$, the restrictions to $\Sigma_{\sigma'}$ satisfy $f \in H^1(\Sigma_{\sigma'}) \cap H^\infty(\Sigma_{\sigma'})$ and $g \in H^1(\Sigma_{\sigma'})$. By the resolvent identity we obtain

$$\begin{aligned} f(A)g(A) &= \left(\frac{1}{2\pi i}\right)^2 \left(\int_{\partial\Sigma_\nu} f(\mu)R(\mu, A) d\mu\right) \left(\int_{\partial\Sigma_{\nu'}} g(\lambda)R(\lambda, A) d\lambda\right) \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Sigma_{\nu'}} \int_{\partial\Sigma_\nu} \frac{f(\mu)g(\lambda)}{\lambda - \mu} [R(\mu, A) - R(\lambda, A)] d\mu d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\nu'}} g(\lambda) \left[\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \frac{f(\mu)}{\mu - \lambda} d\mu \right] R(\lambda, A) d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(\mu) \left[\frac{1}{2\pi i} \int_{\partial\Sigma_{\nu'}} \frac{g(\lambda)}{\lambda - \mu} d\lambda \right] R(\mu, A) d\mu \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\nu'}} g(\lambda) f(\lambda) R(\lambda, A) d\lambda = (fg)(A). \end{aligned}$$

Here we used that

$$\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \frac{f(\mu)}{\mu - \lambda} d\mu = f(\lambda)$$

by Cauchy's formula and

$$\frac{1}{2\pi i} \int_{\partial \Sigma_{\nu'}} \frac{g(\lambda)}{\lambda - \mu} d\lambda = 0$$

by Cauchy's theorem since $\lambda \mapsto g(\lambda)(\lambda - \mu)^{-1}$ is holomorphic on $\Sigma_{\nu'}$.

The convergence property (10.10) follows from the dominated convergence theorem. $\square$

Proposition 10.2.3. *Let A be a sectorial operator and let*

$$\zeta_{\lambda_1, \dots, \lambda_n}(z) := z \prod_{j=1}^n \frac{1}{\lambda_j - z},$$

where $n \geq 2$ and $\lambda_j \neq 0$ satisfies $|\arg(\lambda_j)| > \omega(A)$, $j = 1, \dots, n$. Then

$$\zeta_{\lambda_1, \dots, \lambda_n}(A) = A \prod_{j=1}^n R(\lambda_j, A).$$

Proof. By the convergence property of the Dunford calculus we may assume that $\lambda_i \neq \lambda_j$ for all $i \neq j$. Let $\omega(A) < \nu < \min_{1 \leq j \leq n} |\arg(\lambda_j)|$ and put $\Sigma_\nu(\varepsilon) := \Sigma_\nu \cup B(0, \varepsilon)$ with $0 < \varepsilon < \min_{1 \leq j \leq n} |\lambda_j|$. Set $\psi(z) := \prod_{j=1}^n \frac{1}{\lambda_j - z}$. By Cauchy's theorem,

$$\zeta_{\lambda_1, \dots, \lambda_n}(A)x = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \psi(z) z R(z, A) dz = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} \psi(z) z R(z, A) dz.$$

Writing $zR(z, A) = I + AR(z, A)$ and applying Cauchy's theorem to the function ψ and the residue theorem to the function $\psi(\cdot)AR(\cdot, A)$ (which has poles at $\lambda_1, \dots, \lambda_n$), we obtain

$$\begin{aligned} \zeta_{\lambda_1, \dots, \lambda_n}(A)x &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} \psi(z) dz + \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} \psi(z) AR(z, A) dz \\ &= 0 + \sum_{k=1}^n \prod_{j \neq k} \frac{1}{\lambda_j - \lambda_k} AR(\lambda_k, A) = A \prod_{j=1}^n R(\lambda_j, A), \end{aligned}$$

where the latter follows by repeated application of the resolvent identity. $\square$

As a first application of the part of the lemma we continue with a very useful formula that will play an important role in the developments in later sections, the so-called Calderón reproducing formula. The second part of the proof uses the following elementary lemma, which will also be useful at other places.

Lemma 10.2.4. *Let $\sigma \in (0, \pi)$. For every $z \in \Sigma_\sigma$ we have*

$$|1 + z| \geq (|z| \vee 1) \sin(\sigma \vee \frac{\pi}{2}) = \begin{cases} |z| \vee 1, & \text{if } \sigma \in (0, \frac{1}{2}\pi], \\ (|z| \vee 1) \sin(\sigma), & \text{if } \sigma \in (\frac{1}{2}\pi, \pi). \end{cases}$$

Proof. Let $z = re^{i\nu}$ with $r > 0$ and $\nu \in (-\sigma, \sigma)$. If $|\nu| \leq \frac{1}{2}\pi$, then

$$|1 + z| = \left((1 + r^2 + 2r \cos(\nu))^{1/2}\right) \geq (1 + r^2)^{1/2} \geq r \vee 1.$$

This estimate gives the required estimate if $0 < \sigma \leq \frac{1}{2}\pi$.

Next consider the case $\frac{1}{2}\pi < \sigma < \pi$ and $\frac{1}{2}\pi < |\nu| < \sigma$. If $r \geq 1$, then

$$|1 + z| \geq |\Im(1 + z)| = r|\sin(\nu)| \geq r|\sin(\sigma)|.$$

If $r \in (0, 1)$, then

$$|1 + z|^2 = 1 + r^2 + 2r \cos(\nu) \geq 1 - \cos^2(\nu) = \sin^2(\nu) \geq \sin^2(\sigma).$$

This proves the remaining case of the estimate. $\square$

Proposition 10.2.5 (Calderón reproducing formula I). *Let A be a sectorial operator, let $\omega(A) < \sigma < \pi$, and assume $f \in H^1(\Sigma_\sigma)$ satisfies*

$$\int_0^\infty f(t) \frac{dt}{t} = 1.$$

Then for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have

$$\int_0^\infty f(tA)x \frac{dt}{t} = x,$$

the integral being convergent as an improper integral in X . If $z \mapsto \log(z)f(z)$ belongs to $H^1(\Sigma_\sigma)$, the same conclusion holds for all $x \in \overline{\mathcal{D}(A)} \cap \mathcal{R}(A)$.

The growth condition in second part of the statement holds in particular if

$$|f(z)| \lesssim \min\{|z|^{-\alpha}, |z|^\alpha\}$$

for some $\alpha > 0$ and all $z \in \Sigma_\sigma$. The additional assumption on f can be avoided if A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus (see Proposition 10.2.15 below).

Proof. Fix an arbitrary $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. By analytic continuation the identity $\int_0^\infty f(tz) \frac{dt}{t} = 1$ extends to $z \in \Sigma_\sigma$. Put

$$\varphi_{a,b}(z) := \int_a^b f(z t) \frac{dt}{t}.$$

This function is holomorphic on Σ_σ . For $|\nu| < \sigma$ we use Fubini's theorem and a substitution to estimate

$$\int_0^\infty |\varphi_{a,b}(se^{i\nu})| \frac{ds}{s} \leq \int_a^b \int_0^\infty |f(tse^{i\nu})| \frac{ds}{s} \frac{dt}{t} \leq \log(b/a) \|f\|_{H^1(\Sigma_\sigma)}.$$

Therefore $\varphi_{a,b} \in H^1(\Sigma_\sigma)$. Clearly, we have $\varphi_{a,b}(z) \rightarrow 1$ as $a \downarrow 0$ and $b \rightarrow \infty$. Let $\omega(A) < \nu < \sigma$. By Fubini's theorem,

$$\begin{aligned}
\varphi_{a,b}(A)x &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \varphi_{a,b}(z)R(z, A)x \, dz \\
&= \frac{1}{2\pi i} \int_a^b \int_{\partial\Sigma_\nu} f(tz)R(z, A)x \, dz \frac{dt}{t} = \int_a^b f(tA)x \frac{dt}{t}.
\end{aligned} \tag{10.11}$$

We recall from Propositions 10.1.8 and 10.2.3 that $\mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{R}(\zeta(A))$ with $\zeta(A) = A(I + A)^{-2}$, so we may write $x = \zeta(A)y$ for some $y \in X$. By (10.11) and Theorem 10.2.2 we then obtain

$$\int_a^b f(tA)x \frac{dt}{t} = \varphi_{a,b}(A)\zeta(A)y = (\varphi_{a,b}\zeta)(A)y \rightarrow \zeta(A)y = x$$

as $a \downarrow 0$ and $b \rightarrow \infty$. This completes the proof of the first assertion.

Assume now that $z \mapsto \log(z)f(z)$ belongs to $H^1(\Sigma_\sigma)$. We claim that

$$\sup_{0 < a \leq b < \infty} \left\| \int_a^b f(tA) \frac{dt}{t} \right\| \leq C_{f,\sigma}, \tag{10.12}$$

with an explicit bound on the constant $C_{f,\sigma}$. The improper convergence of $\int_0^\infty f(tA)x \frac{dt}{t}$ for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ then follows from the first part of the proof and an approximation argument.

For all $z \in \Sigma_\sigma$,

$$\begin{aligned}
\varphi_{a,b}(z) &= \int_0^b f(tz) \frac{dt}{t} - \int_0^a f(tz) \frac{dt}{t} \\
&= \int_0^1 f(tbz) \frac{dt}{t} - \int_0^1 f(taz) \frac{dt}{t} =: h(bz) - h(az)
\end{aligned}$$

with $h(z) := \int_0^1 f(tz) \frac{dt}{t}$. Set $g(z) := \int_1^\infty f(tz) \frac{dt}{t}$ and note that

$$\psi(z) := h(z) - \frac{z}{1+z} = -g(z) + \frac{1}{1+z}, \quad z \in \Sigma_\sigma.$$

We claim that $\psi \in H^1(\Sigma_\sigma)$. To show this we use both expressions for ψ on different parts of the H^1 -norm. We have, for $0 < |\nu| < \sigma$,

$$\int_0^1 |\psi(se^{i\nu})| \frac{ds}{s} \leq \int_0^1 |h(se^{i\nu})| \frac{ds}{s} + \int_0^1 \frac{1}{|1 + se^{i\nu}|} ds.$$

Using the inequality $|1 + se^{i\nu}| \geq \sin(\sigma \vee \frac{1}{2}\pi)$ from Lemma 10.2.4, the second integral on the right-hand side is bounded by $1/\sin(\sigma \vee \frac{1}{2}\pi)$. For the first integral we note that

$$\int_0^1 |h(se^{i\nu})| \frac{ds}{s} \leq \int_0^1 \int_0^1 |f(ste^{i\nu})| \frac{dt}{t} \frac{ds}{s}$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 |f(ste^{i\nu})| \frac{ds}{s} \frac{dt}{t} \\
&= \int_0^1 \int_0^s |f(te^{i\nu})| \frac{ds}{s} \frac{dt}{t} \\
&= \int_0^1 \int_t^1 |f(te^{i\nu})| \frac{ds}{s} \frac{dt}{t} = \int_0^1 |f(te^{i\nu}) \log(t)| \frac{dt}{t}.
\end{aligned}$$

On the interval $(1, \infty)$ we have

$$\int_1^\infty |\psi(se^{i\nu})| \frac{ds}{s} \leq \int_1^\infty |g(se^{i\nu})| \frac{dt}{t} + \int_1^\infty \frac{1}{|1 + se^{i\nu}|} \frac{ds}{s}.$$

Using again that $|1 + se^{i\nu}| \geq s \sin(\sigma \vee \frac{1}{2}\pi)$, the latter integral is bounded by $1/\sin(\sigma \vee \frac{1}{2}\pi)$. In the same way as before we see that

$$\int_1^\infty |g(se^{i\nu})| \frac{dt}{t} \leq \int_1^\infty |f(te^{i\nu}) \log(t)| \frac{dt}{t}.$$

It follows that $\psi \in H^1(\Sigma_\sigma)$ with

$$\|\psi\|_{H^1(\Sigma_\sigma)} \leq \|z \mapsto f(z) \log(z)\|_{H^1(\Sigma_\sigma)} + C_\sigma,$$

where $C_\sigma = 2/\sin(\sigma \vee \frac{1}{2}\pi)$.

Let $\omega(A) < \nu < \sigma$. By the above estimates, for all $t > 0$ we have

$$\begin{aligned}
\|\psi(tA)\| &\leq \frac{1}{2\pi} \int_{\partial\Sigma_\nu} |\psi(tz)| \|zR(z, A)\| \frac{|dz|}{|z|} \\
&\leq \frac{M_{\nu, A}}{2\pi} \int_{\partial\Sigma_\nu} |\psi(z)| \frac{|dz|}{|z|} \leq \frac{M_{\nu, A}}{2\pi} (\|z \mapsto f(z) \log(z)\|_{H^1(\Sigma_\sigma)} + C_\sigma)
\end{aligned}$$

Since

$$\varphi_{a,b}(z) = h(bz) - h(az) = \psi(bz) - \psi(az) + \frac{(b-a)z}{(1+bz)(1+az)},$$

we can use the linearity of the Dunford calculus, Proposition 10.2.3 and the resolvent identity to obtain

$$\begin{aligned}
\varphi_{a,b}(A) &= \psi(bA) - \psi(aA) + (b-a)A(1+bA)^{-1}(1+aA)^{-1} \\
&= \psi(bA) - \psi(aA) + (1+aA)^{-1} - (1+bA)^{-1}.
\end{aligned}$$

Hence by the previous estimates (and taking the infimum over all ν),

$$\begin{aligned}
\|\varphi_{a,b}(A)\| &\leq \|\psi(bA)\| + \|\psi(aA)\| + \|(1+bA)^{-1}\| + \|(1+aA)^{-1}\| \\
&\leq 2M_{\sigma, A} \left(\frac{\|z \mapsto f(z) \log(z)\|_{H^1(\Sigma_\sigma)} + C_\sigma}{2\pi} + 1 \right).
\end{aligned}$$

This completes the proof of (10.12). $\square$

As an application of the second part of Proposition 10.2.3 we describe a useful approximation procedure.

Proposition 10.2.6. *For $n \geq 1$ the functions*

$$\zeta_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}$$

belong to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ for all $0 < \sigma < \pi$, and they satisfy

$$\|\zeta_n\|_{H^\infty(\Sigma_\sigma)} \leq \frac{1}{\sin(\sigma \vee \frac{1}{2}\pi)}.$$

In particular, for $0 < \sigma \leq \frac{1}{2}\pi$ we have $\|\zeta_n\|_{H^\infty(\Sigma_\sigma)} \leq 1$. Moreover, for any sectorial operator A ,

(1) we have

$$\zeta_n(A) = n(n+A)^{-1} - \frac{1}{n} \left(\frac{1}{n} + A \right)^{-1}$$

and for all $\sigma > \omega(A)$

$$\sup_{n \geq 1} \|\zeta_n(A)\| \leq 2M_{\sigma,A};$$

(2) for all $x \in X$ we have $\zeta_n(A)x \in D(A) \cap R(A)$;

(3) for all $x \in \overline{D(A)} \cap R(A)$ we have

$$\lim_{n \rightarrow \infty} \zeta_n(A)x = x.$$

Proof. That $\zeta_n \in H^1(\Sigma_\sigma)$ is clear by writing

$$\zeta_n(z) = \frac{(n^2 - 1)z}{(n+z)(1+nz)}. \quad (10.13)$$

To obtain the bound for their norms in $H^\infty(\Sigma_\sigma)$ write

$$\zeta_n(z) = \frac{n^2 - 1}{n^2} \frac{1}{(1 + \frac{z}{n})(1 + \frac{1}{nz})}.$$

For $z = re^{i\nu}$ with $r > 0$ and $|\nu| < \sigma$, we have

$$\begin{aligned} w &:= n^2 \left(1 + \frac{z}{n}\right) \left(1 + \frac{1}{nz}\right) = n^2 + 1 + nre^{i\nu} + \frac{n}{r}e^{-i\nu} \\ &= n^2 + 1 + n\left(r + \frac{1}{r}\right) \cos(\nu) + in\left(r - \frac{1}{r}\right) \sin(\nu). \end{aligned}$$

Therefore, if $\nu \in [0, \pi/2]$, then $\cos(\nu) \geq 0$ and hence $|\zeta(z)| \leq \frac{n^2-1}{n^2+1} \leq 1$. It remains to check that for $\nu \in [\pi/2, \pi]$, we have $|\zeta(z)| \leq 1/\sin(\nu)$. The latter holds if and only if

$$|w|^2 \geq (n^2 - 1)^2 \sin^2(\nu).$$

The latter inequality indeed holds, for it is elementary to verify that

$$|w|^2 - (n^2 - 1)^2 \sin^2(\nu) = \frac{1}{r^2} \left((n^2 + 1)r \cos(\nu) + n(r^2 + 1) \right)^2 \geq 0.$$

(1): The identity for the operators $\zeta_n(A)$ follows from (10.13) and Proposition 10.2.3, and their uniform boundedness with constant $2M_{\sigma,A}$ from the sectoriality of A .

(2): That $\zeta_n(A)x \in \mathcal{D}(A)$ is immediate from the identity in (1), and that $\zeta_n(A)x \in \mathcal{R}(A)$ follows by rewriting

$$\zeta_n(A) = (n^2 - 1)A(n + A)^{-1}(1 + nA)^{-1}.$$

(3): This is immediate from the assertions in part (1) and the convergence results of (10.4) and (10.5). $\square$

The analytic semigroup generated by a densely defined sectorial operator

It is shown in Theorem G.5.2 that if A is a densely defined sectorial operator on a Banach space X , then $-A$ generates a bounded analytic C_0 -semigroup $(S(t))_{t \geq 0}$ which is given by the inverse Laplace transform representation

$$S(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{tz} R(z, -A)x \, dz, \quad t > 0, \quad x \in X,$$

where Γ is the upwards oriented contour in the left-half plane specified in the theorem.

Here we wish to point out that the operators $S(t)$ can also be constructed directly from the Dunford calculus of A . This is not immediately obvious, as one cannot define them as $S(t) = \exp(-tA)$ using the exponential function $\exp(-tz)$: this function does not belong to $H^1(\Sigma_\sigma)$ for any $0 < \sigma < \pi$. This defect can be remedied by compensating the non-integrability near 0 by an additional additive term

$$\exp(-tz) = \left(\exp(-tz) - \frac{1}{1 + tz} \right) + \frac{1}{1 + tz}$$

and interpreting $\frac{1}{1+tz}(A)$ as $(I + tA)^{-1}$.

Proposition 10.2.7. *Let A be a densely defined sectorial operator satisfying $\omega(A) < \frac{1}{2}\pi$, let $\omega(A) < \sigma < \frac{1}{2}\pi$ and $\eta \in (0, \frac{1}{2}\pi - \sigma)$, and denote by $(S(\zeta))_{\zeta \in \Sigma_\eta}$ the bounded analytic C_0 -semigroup generated by $-A$. Then for all $z \in \Sigma_\eta$ the following identity holds:*

$$S(\zeta) - (1 + \zeta A)^{-1} = f_\zeta(A),$$

where

$$f_\zeta(z) = \exp(-\zeta z) - \frac{1}{1 + \zeta z}.$$

Observe that $z \mapsto \exp(-z) - \frac{1}{1+z}$ belongs to $H^1(\Sigma_{\sigma+\eta})$, so f_ζ belongs to $H^1(\Sigma_\sigma)$ and $f_\zeta(A)$ can be defined through the Dunford calculus of A .

Proof. It suffices to prove the identity for $\zeta = t > 0$, since then the result for $\zeta \in \Sigma_\eta$ follows by analytic continuation. Fix $t > 0$ and $0 < \varepsilon < 1/t$, and put $\Sigma_\eta(\varepsilon) := \Sigma_\eta \cup B(0, \varepsilon)$. By Cauchy's theorem, the path of integration in the Dunford integral may be changed from $\partial\Sigma_\eta$ to $\partial\Sigma_\eta(\varepsilon)$ to obtain

$$\begin{aligned} f_t(A) &= \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} f_t(z) R(z, A) \, dz \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} e^{-tz} R(z, A) \, dz - \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} \frac{1}{1+tz} R(z, A) \, dz. \end{aligned}$$

After substituting $z \mapsto -z$, the first integral is seen to coincide with the inverse Laplace transform representation of $S(t)$ of Theorem G.5.2 (applied to $-A$). To evaluate the second integral we use the holomorphy of $z \mapsto R(z, A)x$ and Cauchy's theorem:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} \frac{1}{1+tz} R(z, A) \, dz &= -\frac{1}{2\pi i t} \int_{\Gamma} \frac{R(z, A)}{-t^{-1} - z} \, dz \\ &= -t^{-1} R(-t^{-1}, A) = (I + tA)^{-1}. \end{aligned}$$

This proves the identity $f_t(A) = S(t) - (I + tA)^{-1}$. $\square$

It is worth mentioning that for densely defined sectorial operators A satisfying $\omega(A) < \frac{1}{2}\pi$, the convergence assertions in parts (1) and (2) of Proposition 10.1.8 have counterparts in terms of the semigroup $(S(t))_{t \geq 0}$ generated by $-A$. As a rule of thumb, for $t \rightarrow \infty$ (resp. for $t \downarrow 0$) the limiting behaviour of $S(t)$ is the same as that for $A(t+A)^{-1}$ $t \rightarrow \infty$ (resp. for $t \downarrow 0$), and also the same as that for $t(t+A)^{-1}$ for $t \downarrow 0$ (resp. for $t \rightarrow \infty$). Let us illustrate this by proving one of these correspondences: we will show that

$$\text{for all } x \in \overline{\mathcal{R}(A)} \text{ we have } \lim_{t \rightarrow \infty} S(t)x = 0.$$

By the inverse Laplace transform representation,

$$S(t)x = \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} e^{tz} (z+A)^{-1} x \, dz = \frac{1}{2\pi i} \int_{\partial\Sigma_\eta(\varepsilon)} \frac{e^z}{z} \left[\frac{z}{t} \left(\frac{z}{t} + A \right)^{-1} \right] x \, dz,$$

where $\Sigma_\eta(\varepsilon)$ is as before. In the second identity we used Cauchy's theorem to justify that we did not change the path of integration. By repeating the proof of Proposition 10.1.7(2) *verbatim*, $\frac{z}{t} \left(\frac{z}{t} + A \right)^{-1}$ tends to 0 strongly as $t \rightarrow \infty$ on $\overline{\mathcal{R}(A)}$. Therefore the claim follows by dominated convergence.

We can push this idea a little further. Writing (cf. Proposition G.2.3) $S(t)x - x = -\int_0^t AS(s)x \, ds$, for $x \in \overline{\mathcal{R}(A)}$ we may pass to the limit $t \rightarrow \infty$ to obtain a special case of the Calderón reproducing formula (Proposition 10.2.5):

$$x = \lim_{t \rightarrow \infty} x - S(t)x = \int_0^\infty AS(s)x \, ds = \int_0^\infty f(sA)x \frac{ds}{s},$$

for $f(z) = z \exp(-z)$, with improper convergence of the integral. To justify the last step in this computation we use the following fact which, for later reference, we state as a lemma:

Lemma 10.2.8. *Let A be a densely defined sectorial operator and let $f(z) = z \exp(-z)$. Then for all $t > 0$ we have $f(tA) = tAS(t)$.*

Proof. Fixing $\omega(A) < \nu < \frac{1}{2}\pi$ and using the notation as before, we compute

$$\begin{aligned} f(tA) &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} z e^{-z} R(z, tA) \, dz \\ &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} t z e^{-tz} R(z, A) \, dz \\ &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} t e^{-tz} A R(z, A) \, dz + \frac{1}{2\pi i} \int_{\partial \Sigma_\nu(\varepsilon)} t e^{-tz} \, dz \\ &= \frac{1}{2\pi i} A \int_{\partial \Sigma_\nu(\varepsilon)} t e^{-tz} R(z, A) \, dz \\ &= tAS(t), \end{aligned}$$

where the last step follows from Theorem G.5.2 after the substitution $z \mapsto -z$. In the computation Cauchy's theorem was used twice, first to see that one integral vanishes, and then to change the path of integration to the one used in Theorem G.5.2. Hille's theorem (Theorem 1.2.4) was used to pull A out of the integral. $\square$

The technique used in Proposition 10.2.7 is not limited the example treated here, but it will not pursue it any further. Instead, Volume III will include an account of the so-called *extended Dunford calculus*, which systematically develops an idea introduced in (10.14) and allows the treatment of a wider ranges of examples. For now we content ourselves with a variation on this theme which provides a formula for the operators $R(\mu, A)$ in terms of a Dunford integral.

Example 10.2.9. Let A be a sectorial operator and let $\omega(A) < \sigma < \pi$. Then for all $\mu \notin \overline{\Sigma_\sigma}$ for all $x \in \mathcal{D}(A)$ we have

$$\frac{1}{2\pi i} \int_{\partial \Sigma_\sigma} (\mu - z)^{-1} R(z, A)x \, dz = R(\mu, A)x.$$

To see that the integral is absolutely convergent near the origin, we write $x = Ay$ with $y \in \mathcal{D}(A)$ and use that $R(z, A)x = AR(z, A)y = zR(z, A)y - y$ remains bounded near 0. Defining $\Sigma_\nu(\varepsilon)$ as before, from Cauchy's integral theorem and the holomorphy of $z \mapsto R(z, A)x$ we find

$$\frac{1}{2\pi i} \int_{\partial \Sigma_\sigma} \frac{R(z, A)x}{\mu - z} \, dz = \frac{1}{2\pi i} \int_{\partial \Sigma_\sigma(\varepsilon)} \frac{R(z, A)x}{\mu - z} \, dz = R(\mu, A)x.$$

10.2.b The H^∞ -calculus

We now restrict the Dunford calculus to functions in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and address the problem of extending this restricted calculus to general functions in $H^\infty(\Sigma_\sigma)$. This is a non-trivial problem, if only because $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ fails to be dense in $H^\infty(\Sigma_\sigma)$, but more profoundly, because the Dunford integrals become singular for functions in $H^\infty(\Sigma_\sigma)$.

Definition 10.2.10 (Operators with a bounded H^∞ -calculus). *Let A be a sectorial operator on X and let $\omega(A) < \sigma < \pi$. The operator A is said to have a bounded $H^\infty(\Sigma_\sigma)$ -calculus if there exists a constant $C \geq 0$ such that*

$$\|f(A)\| \leq C\|f\|_\infty, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma).$$

We define

$$\omega_{H^\infty}(A) := \inf \{ \omega(A) < \sigma < \pi : A \text{ has a bounded } H^\infty(\Sigma_\sigma)\text{-calculus} \}.$$

Suppose now that A is a sectorial operator on X and let $f \in H^\infty(\Sigma_\sigma)$ with $\omega(A) < \sigma < \pi$. By Propositions 10.1.8 and 10.2.3, for the function $\zeta(z) := z(1+z)^{-2}$ we have $\zeta(A) = A(I+A)^{-2}$ and $D(A) \cap R(A) = R(\zeta(A))$. If $x \in D(A) \cap R(A)$, say $x = \zeta(A)y$ with $y \in X$, define

$$\Psi_A(f)x := (f\zeta)(A)y \tag{10.14}$$

using the Dunford calculus of A applied to $f\zeta \in H^1(\Sigma_\sigma)$. To see that this is well defined suppose that $\zeta(A)y = 0$. Then $(I+A)^{-1}[(1+A)^{-1}y - y] = -A(I+A)^{-2}y = 0$, so $(1+A)^{-1}y = y$, and by the resolvent identity this implies $R(z, A)y = z^{-1}y$. Then, with $\omega(A) < \nu < \sigma$,

$$(f\zeta)(A)y = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z)\zeta(z)R(z, A)y \, dz = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \frac{f(z)}{(1+z)^2} y \, dz = 0$$

by Cauchy's theorem.

For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ we have $\Psi_A(f)x = (f\zeta)(A)y = f(A)\zeta(A)y = f(A)x$ by the multiplicativity of the Dunford calculus. This shows that

$$\Psi_A(f) = f(A) \text{ on } D(A) \cap R(A). \tag{10.15}$$

The next proposition shows that whether or not A has a bounded H^∞ -calculus is completely determined by the part of A in $\overline{D(A)} \cap R(A)$.

Proposition 10.2.11. *Let A be a sectorial operator on X and let $\omega(A) < \sigma < \pi$. Let $C > 0$. The following assertions are equivalent:*

- (1) *A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus, i.e., there exists a constant $C \geq 0$ such that for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$,*

$$\|f(A)x\| \leq C\|f\|_\infty\|x\|, \quad x \in X;$$

(2) there exists a constant $C \geq 0$ such that for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$,

$$\|f(A)x\| \leq C\|f\|_\infty\|x\|, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A);$$

(3) there exists a constant $C \geq 0$ such that for all $f \in H^\infty(\Sigma_\sigma)$,

$$\|\Psi_A(f)x\| \leq C\|f\|_\infty\|x\|, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A).$$

If $C_{(1)}$, $C_{(2)}$, and $C_{(3)}$ denote the respective best constants, then $C_{(2)} \leq C_{(1)} \leq 2M_{\sigma,A}C_{(3)}$ and $C_{(3)} \leq C_{(2)}/\sin(\sigma \vee \frac{1}{2}\pi)$, and if $\mathcal{D}(A) \cap \mathcal{R}(A)$ is dense, then $C_{(1)} \leq C_{(3)}$.

Observe that if $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$, then $C_{(3)} \leq C_{(2)}$, and if in addition $\mathcal{D}(A) \cap \mathcal{R}(A)$ is dense then all three constants are equal:

$$C_{(1)} = C_{(2)} = C_{(3)}.$$

Proof. (1) $\Rightarrow$ (2): This holds trivially, with bound $C_{(2)} \leq C_{(1)}$.

(2) $\Rightarrow$ (3): Let $f \in H^\infty(\Sigma_\sigma)$ be given and let ζ_n be the functions of Proposition 10.2.6. Then $\zeta_n f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. Using the definition (10.14), the convergence property of the Dunford calculus, its multiplicativity, the assumption in (2), and the H^∞ -bound for the ζ_n of Proposition 10.2.6 we obtain, for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, say $x = \zeta(A)y$,

$$\begin{aligned} \|\Psi_A(f)x\| &= \|(f\zeta)(A)y\| = \lim_{n \rightarrow \infty} \|(\zeta_n f \zeta)(A)y\| \\ &= \lim_{n \rightarrow \infty} \|(\zeta_n f)(A)\zeta(A)y\| \leq C \limsup_{n \rightarrow \infty} \|\zeta_n f\|_\infty \|\zeta(A)y\| \\ &\leq \frac{C}{\sin(\sigma \vee \frac{1}{2}\pi)} \|f\|_\infty \|x\|. \end{aligned}$$

This proves that (3) holds, with bound $C_{(3)} \leq C_{(2)}/\sin(\sigma \vee \frac{1}{2}\pi)$.

(3) $\Rightarrow$ (1): For any $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $x \in X$ we have $\zeta_n(A)x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and therefore by the convergence property of the Dunford calculus, its multiplicativity, (10.15), and the assumption in (3),

$$\begin{aligned} \|f(A)x\| &= \lim_{n \rightarrow \infty} \|(\zeta_n f)(A)x\| = \lim_{n \rightarrow \infty} \|f(A)\zeta_n(A)x\| \\ &= \lim_{n \rightarrow \infty} \|\Psi_A(f)\zeta_n(A)x\| \leq C\|f\|_\infty \liminf_{n \rightarrow \infty} \|\zeta_n(A)x\|. \end{aligned}$$

If $\mathcal{D}(A) \cap \mathcal{R}(A)$ is dense, for $x \in X$ we have $\lim_{n \rightarrow \infty} \zeta_n(A)x = x$ by the convergence assertion in Proposition 10.2.6; this gives (1) with bound $C_{(1)} \leq C_{(3)}$. In general, all we can say is that $\|\zeta_n(A)x\| \leq 2M_{\sigma,A}C\|x\|$ using the bound on $\zeta_n(A)$ of Proposition 10.2.6; this gives (1) with bound $C_{(1)} \leq 2M_{\sigma,A}C_{(3)}$. $\square$

Suppose now that A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus. We claim that for all $f \in H^\infty(\Sigma_\sigma)$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have $\Psi_A(f)x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. To see

this it suffices to consider $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, say $x = \zeta(A)y$. Then, arguing as in the above proof,

$$\Psi_A(f)x = (f\zeta)(A)y = \lim_{n \rightarrow \infty} (\zeta_n f \zeta)(A)y = \lim_{n \rightarrow \infty} \zeta(A)(\zeta_n f)(A)y$$

with $\zeta(A)(\zeta_n f)(A)y \in \mathcal{R}(\zeta(A)) = \mathcal{D}(A) \cap \mathcal{R}(A)$. This proves the claimed invariance.

We may now define a bounded operator

$$f(A) : \overline{\mathcal{D}(A) \cap \mathcal{R}(A)} \rightarrow \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$$

by

$$f(A)x := \Psi_A(f)x, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A).$$

By (10.15), for $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ this definition is consistent with the Dunford calculus.

Definition 10.2.12 (The H^∞ -calculus). *Let A be a sectorial operator admitting a bounded $H^\infty(\Sigma_\sigma)$ -calculus. The mapping*

$$f \mapsto f(A)$$

from $H^\infty(\Sigma_\sigma)$ to $\mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})$ defined above is called the $H^\infty(\Sigma_\sigma)$ -calculus of A .

By Proposition 10.2.11 we have the norm estimate

$$\|f(A)\|_{\mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})} \leq M_{\sigma,A}^\infty \|f\|_\infty, \quad f \in H^\infty(\Sigma_\sigma),$$

where $M_{\sigma,A}^\infty := C_{(3)}$ is the best constant in part (3) of the proposition. This constant will be referred to as the *boundedness constant* of the $H^\infty(\Sigma_\sigma)$ -calculus of A . Note that we have already defined the angle $\omega_{H^\infty}(A)$ of the $H^\infty(\Sigma_\sigma)$ -calculus through Definition 10.2.10; Proposition 10.2.11 shows that we could equivalently have defined this angle using Definition 10.2.12.

The next theorem describes some elementary properties of the H^∞ -calculus.

Theorem 10.2.13. *Let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\sigma)$ -calculus. The mapping*

$$f \mapsto f(A)$$

from $H^\infty(\Sigma_\sigma)$ to $\mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})$ is linear and multiplicative and satisfies

$$\mathbf{1}(A)x = x \quad \text{and} \quad r_\mu(A)x = R(\mu, A)x, \quad x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)},$$

where $r_\mu(z) = (\mu - z)^{-1}$ with $\mu \notin \overline{\Sigma_\sigma}$. Furthermore, if the functions $f_n, f \in H^\infty(\Sigma_\sigma)$ are uniformly bounded and satisfy $f_n(z) \rightarrow f(z)$ for all $z \in \Sigma_\sigma$, then

$$\lim_{n \rightarrow \infty} f_n(A)x = f(A)x, \quad x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}. \quad (10.16)$$

Proof. The linearity and multiplicativity are consequences of the convergence property, since it allows us to approximate functions f in $H^\infty(\Sigma_\sigma)$ by functions $f_n = \zeta_n f$ in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, where the functions ζ_n are as in Proposition 10.2.6, and apply the linearity and multiplicativity of the Dunford calculus.

For $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ the identity $\mathbf{1}(A)x = x$ is clear from (10.14). Similarly, writing $x = \zeta(A)y$, by (10.14) we have $r_\mu(A)x = (r_\mu\zeta)(A)y$. If we can prove that $(r_\mu\zeta)(A) = AR(\mu, A)(I + A)^{-2}$ the identity $r_\mu(A)x = R(\mu, A)x$ follows from it since $x = \zeta(A)y = A(I + A)^{-2}y$. But this identity follows from Proposition 10.2.3.

Both identities $\mathbf{1}(A)x = x$ and $r_\mu(A)x = R(\mu, A)x$ extend to $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ by the boundedness of the operators involved. Note that the identity for the resolvent is consistent with the formula in Example 10.2.9.

We now turn to the proof of the convergence property. Using first (10.14), then the convergence property of the Dunford calculus, and then again (10.14), for all $x = \zeta(A)y \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we obtain

$$\lim_{n \rightarrow \infty} f_n(A)x = \lim_{n \rightarrow \infty} (f_n\zeta)(A)y = (f\zeta)(A)y = f(A)x.$$

This gives the desired convergence on the dense subspace $\mathcal{D}(A) \cap \mathcal{R}(A)$. Since the operators $f_n(A)$ are uniformly bounded (as the functions f_n are uniformly bounded in $H^\infty(\Sigma_\sigma)$), the result follows from this. $\square$

Theorem 10.2.14 (Uniqueness of the H^∞ -calculus). *Let A be a sectorial operator on X and let $\omega(A) < \sigma < \pi$. Suppose $\Psi : H^\infty(\Sigma_\sigma) \rightarrow \mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})$ is bounded, linear and multiplicative, satisfies*

$$\Psi r_\mu = R(\mu, A)$$

for all functions $r_\mu(z) = (\mu - z)^{-1}$ with $\mu \notin \overline{\Sigma_\sigma}$, and has the following convergence property:

if f_n, f are functions in $H^\infty(\Sigma_\sigma)$ satisfying $f_n \rightarrow f$ pointwise on Σ_σ and $\sup_n \|f_n\|_\infty < \infty$, then $\Psi(f_n)x \rightarrow \Psi(f)x$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Then:

- (1) A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus;
- (2) $\Psi f = f(A)$ as operators in $\mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})$ for all $f \in H^\infty(\Sigma_\sigma)$.

Proof. First consider a function $f \in H^1(\Sigma_\vartheta) \cap H^\infty(\Sigma_\vartheta)$ with $\sigma < \vartheta < \pi$. We can interpret Cauchy's formula as a Bochner integral in $H^\infty(\Sigma_\sigma)$:

$$f(\cdot) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\sigma'}} \frac{f(\mu)}{\mu - \cdot} d\mu = \int_{\partial \Sigma_{\sigma'}} h(\mu) r_\mu(\cdot) d\mu$$

where $\sigma < \sigma' < \vartheta$, the function $\mu \mapsto r_\mu(\cdot) := \mu(\mu - \cdot)^{-1}$ is uniformly bounded and continuous on $\partial \Sigma_{\sigma'}$ as a function with values in $H^\infty(\Sigma_\sigma)$, and $\mu \mapsto$

$h(\mu) := \frac{1}{2\pi i} f(\mu)/\mu$ is integrable on $\partial\Sigma_{\sigma'}$. The convergence property implies that for any $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ the operator $\Psi_x : f \mapsto \Psi(f)x$ is closed as an operator from $H^\infty(\Sigma_\sigma)$ to X . Therefore, by Hille's theorem (Theorem 1.2.4) we may pull Ψ_x through the integral defining $\Psi(f)$:

$$\begin{aligned} \Psi(f)x &= \int_{\partial\Sigma_{\sigma'}} h(\mu) \Psi(r_\mu)x \, d\mu = \int_{\partial\Sigma_{\sigma'}} h(\mu) \mu R(\mu, A)x \, d\mu \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\sigma'}} f(\mu) R(\mu, A)x \, d\mu = f(A)x. \end{aligned}$$

If $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, consider the function $f_n(z) = f(z^{1-\frac{1}{n}})$ which belongs to $H^1(\Sigma_{\sigma\frac{n}{n-1}}) \cap H^\infty(\Sigma_{\sigma\frac{n}{n-1}})$ for $n \geq 2$. We just saw that $\Psi(f_n) = f_n(A)$. Furthermore $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ and $(f_n)_{n \geq 1}$ is uniformly bounded in $H^\infty(\Sigma_\sigma)$. Therefore, the convergence property implies $\Psi(f_n)x \rightarrow \Psi(f)x$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. On the other hand, writing $x = \zeta(A)y \in \mathcal{D}(A) \cap \mathcal{R}(A)$, we find

$$f_n(A)x = f_n(A)\zeta(A)y = (f_n\zeta)(A)y \rightarrow (f\zeta)(A)y = f(A)\zeta(A)y = f(A)x,$$

where we used Theorem 10.2.2. This shows that $f(A) = \Psi(f)$ on $\mathcal{D}(A) \cap \mathcal{R}(A)$, and then on $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ by the boundedness of these operators. As a consequence, for all $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ we obtain

$$\|f(A)x\| = \|\Psi(f)x\| \leq \|\Psi\| \|f\|_\infty \|x\|.$$

By Proposition 10.2.11 this proves (1).

For a general $f \in H^\infty(\Sigma_\sigma)$ there are $f_n \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ with $f_n(\lambda) \rightarrow f(\lambda)$ for $\lambda \in \Sigma_\sigma$ and $\sup_n \|f_n\|_{H^\infty(\Sigma_\sigma)} < \infty$. As both $f \mapsto \Psi f$ and $f \mapsto f(A)$ have the convergence property (the latter in the sense of Theorem 10.2.13), for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we conclude that $\Psi(f)x = \lim_{n \rightarrow \infty} \Psi(f_n)x = \lim_{n \rightarrow \infty} f_n(A)x = f(A)x$, and by a density argument this extends to $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$. This proves (2). $\square$

Assuming the boundedness of the H^∞ -calculus we can eliminate the growth assumption in the second part of the Calderón reproducing formula (Proposition 10.2.5):

Proposition 10.2.15 (Calderón reproducing formula II). *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus, where $\omega(A) < \sigma < \pi$, and suppose that $f \in H^1(\Sigma_\sigma)$ satisfies*

$$\int_0^\infty f(t) \frac{dt}{t} = 1.$$

Then for all $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ we have

$$\int_0^\infty f(tA)x \frac{dt}{t} = x,$$

the integral being convergent as an improper integral in X . Moreover,

$$\sup_{0 < a \leq b < \infty} \left\| \int_a^b f(tA)x \frac{dt}{t} \right\| \leq M_{\sigma,A}^\infty \|f\|_{H^1(\Sigma_\sigma)} \|x\|.$$

Proof. In the first part of Proposition 10.2.5 we have already seen that the reproducing identity holds for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Moreover, as in the proof of the second part of the proposition, the identity extends to $\overline{\mathcal{D}(A)} \cap \mathcal{R}(A)$ if we can prove the uniform boundedness of the operators $x \mapsto \varphi_{a,b}(A)x := \int_a^b f(tA)x \frac{dt}{t}$, $0 < a \leq b < \infty$, on this space. But in view of

$$\|\varphi_{a,b}\|_{H^\infty(\Sigma_\sigma)} = \sup_{z \in \Sigma_\sigma} |\varphi_{a,b}(z)| \leq \sup_{z \in \Sigma_\sigma} \int_0^\infty |f(tz)| \frac{dt}{t} = \|f\|_{H^1(\Sigma_\sigma)},$$

this follows from the boundedness of the $H^\infty(\Sigma_\sigma)$ -calculus. $\square$

We conclude this section with two constructions of new sectorial operators from old. The first describes a simple scaling property that will be frequently used without comment. For $f : \Sigma_\vartheta \rightarrow \mathbb{C}$ and $\mu \in \mathbb{C}$ put

$$f_\mu(z) := f(\mu z), \quad z \in \Sigma_\sigma.$$

Lemma 10.2.16. *Let A be a sectorial operator and let $\omega(A) < \sigma < \vartheta < \pi$ and $\mu \in \Sigma_{\vartheta-\sigma}$. If $f \in H^1(\Sigma_\vartheta)$, then:*

- (1) μA is sectorial with $\omega(\mu A) < \vartheta$;
- (2) $f_\mu \in H^1(\Sigma_\sigma)$ and $f_\mu(A) = f(\mu A)$;
- (3) the mapping $\mu \mapsto f(\mu A)$ is holomorphic from $\Sigma_{\sigma-\vartheta}$ to $\mathcal{L}(\overline{\mathcal{D}(A)} \cap \mathcal{R}(A))$.

Moreover, if A admits a bounded $H^\infty(\Sigma_\sigma)$ -calculus and $f \in H^\infty(\Sigma_\vartheta)$, then:

- (1)' μA admits a bounded H^∞ -calculus with $\omega_{H^\infty}(\mu A) < \vartheta$;
- (2)' $f_\mu \in H^\infty(\Sigma_\sigma)$ and $f_\mu(A) = f(\mu A)$;
- (3)' the mapping $\mu \mapsto f(\mu A)$ is holomorphic from $\Sigma_{\sigma-\vartheta}$ to $\mathcal{L}(\overline{\mathcal{D}(A)} \cap \mathcal{R}(A))$.

Proof. The first two assertions are obvious. For the third, let $\omega(A) < \nu < \sigma$. Let D be a closed disk contained in $\Sigma_{\vartheta-\sigma}$. By Fubini's theorem and Cauchy's theorem, we obtain

$$\int_{\partial D} f(\mu A) d\mu = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \int_{\partial D} f(\mu z) R(z, A) d\mu dz = 0.$$

Now the holomorphy of $\mu \mapsto f(\mu A)$ follows from Morera's theorem.

The proofs of (1)'–(3)' are similar. For (3)' let ζ_n be the functions considered in Proposition 10.2.6. By (3), $\mu \mapsto f(\mu A)\zeta_n(A)$ is holomorphic and thus

$$\int_{\partial D} f(\mu A)\zeta_n(A)x d\mu = 0, \quad x \in \overline{\mathcal{D}(A)} \cap \mathcal{R}(A).$$

By the dominated convergence theorem and Proposition 10.2.6,

$$\int_{\partial D} f(\mu A)x \, d\mu = 0, \quad x \in \overline{D(A) \cap R(A)}.$$

Therefore, the holomorphy of $\mu \mapsto f(\mu A)x$ follows from Morera's theorem. Inspection of the proof of Proposition B.3.1 shows that $\mu \mapsto f(\mu A) \in \mathcal{L}(\overline{D(A) \cap R(A)})$ is continuous in the uniform operator topology. Therefore, Corollary B.3.3 implies that $\mu \mapsto f(\mu A)$ is holomorphic as an operator-valued function. $\square$

The second construction formalises the idea of “substituting” A into an integral expression.

Lemma 10.2.17. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Assume either one of the following:*

- (a) $p = 1$;
- (b) $p = \infty$, A has a bounded H^∞ -calculus, and $\omega_{H^\infty}(A) < \sigma < \pi$.

Let $(S, \mathcal{A}, \mu)$ be a σ -finite measure space. Suppose that $f : S \times \Sigma_\sigma \rightarrow \mathbb{C}$ is a measurable function with the following properties:

- (i) $z \mapsto f(s, z)$ belongs to $H^p(\Sigma_\sigma)$ for all $s \in S$;
- (ii) $\sup_{|\nu| < \sigma} \left\| t \mapsto \int_S |f(s, e^{i\nu}t)| \, d\mu(s) \right\|_{L^p(\mathbb{R}_+, \frac{dt}{t})} < \infty$.

Then the function $g(z) = \int_S f(s, z) \, d\mu(s)$ belongs to $H^p(\Sigma_\sigma)$ and

$$g(A)x = \int_S f(s, A)x \, d\mu(s), \quad \begin{cases} x \in X & \text{in case (a)} \\ x \in \overline{D(A) \cap R(A)} & \text{in case (b)} \end{cases}.$$

Proof. (a): Let D be a closed disk contained in Σ_σ . Applying first Fubini's theorem and then Cauchy's theorem we see that

$$\int_{\partial D} g(\lambda) \, d\lambda = \int_S \left(\int_{\partial D} f(s, \lambda) \, d\lambda \right) d\mu(s) = 0.$$

Hence Morera's theorem implies that g is holomorphic on Σ_σ . Also,

$$\sup_{|\nu| < \sigma} \|g(e^{i\nu} \cdot)\|_{L^p(\mathbb{R}_+, \frac{dt}{t})} \leq \sup_{|\nu| < \sigma} \left\| t \mapsto \int_S |f(s, e^{i\nu}t)| \, d\mu(s) \right\|_{L^p(\mathbb{R}_+, \frac{dt}{t})} < \infty,$$

and thus $g \in H^p(\Sigma_\sigma)$. By Fubini's theorem,

$$\begin{aligned} g(A)x &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g(\lambda) R(\lambda, A)x \, d\lambda \\ &= \int_S \left(\frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(s, \lambda) R(\lambda, A)x \, d\lambda \right) d\mu(s) = \int_S f(s, A)x \, d\mu(s) \end{aligned}$$

for all $x \in X$.

(b): Let ζ_n be the functions considered in Proposition 10.2.6 and put

$$f_n(s, z) := \zeta_n(z)f(s, z), \quad g(z) := \int_S f_n(s, z) d\mu(s).$$

Then $f_n \in H^1(\Sigma_\sigma)$ and hence, by the result of (a),

$$g_n(A)x = \int_S f_n(s, A)x d\mu(s), \quad x \in X.$$

Letting $n \rightarrow \infty$, the required identity follows from Theorem 10.2.13 and the dominated convergence theorem. $\square$

Proposition 10.2.18 (Restrictions). *Let $Y \subseteq X$ be a closed subspace which is invariant under $R(\lambda, A)$ for all $\lambda \in \varrho(A)$. Then the part A_Y of A to Y has a bounded H^∞ -calculus on X , then A_Y has a bounded H^∞ -calculus on Y with $\omega_{H^\infty}(A_Y) \leq \omega_{H^\infty}(A)$, and for all $f \in H^\infty(\Sigma_\sigma)$ with $\omega_{H^\infty}(A) < \sigma < \pi$ we have*

$$f(A_Y)x = f(A)x, \quad x \in Y.$$

Proof. For $\omega(A) < \sigma < \pi$ and $\lambda \in \mathbb{C}\overline{\Sigma_\sigma}$, the restricted resolvent operator $R(\lambda, A)|_Y$ is a two-sided inverse of $\lambda - A_Y$, and therefore $\lambda \in \varrho(A_Y)$ and $R(\lambda, A_Y) = R(\lambda, A)|_Y$. From this, the rest trivially follows. $\square$

It suffices to assume the invariance of Y under the resolvent for one point $\lambda_0 \in \mathbb{C}\overline{\Sigma_\sigma}$. Indeed, for $|\lambda - \lambda_0| < \delta$ with $\delta > 0$ small enough, by the Neumann series argument of Lemma G.1.1 (or repeated use of the resolvent identity) we have $\lambda \in \varrho(A)$ and

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1}.$$

This implies that Y is invariant under $R(\lambda, A)$. The set of all $\lambda \in \varrho(A)$ for which the resolvent leaves Y invariant is therefore open, and evidently it is closed as well. It therefore includes the connected component of $\varrho(A)$ containing λ_0 , which contains $\mathbb{C}\overline{\Sigma_\sigma}$; the latter is all that was used in the proof of the proposition.

Next we will consider the functional calculus for the adjoint operator A^* . A brief introduction to the theory of adjoint operators can be found in Section G.1. If A is a densely defined sectorial operator, then its adjoint A^* is sectorial as well and $\omega(A) = \omega(A^*)$; this is immediate from the fact that $R(\lambda, A^*) = R(\lambda, A)^*$ for all $\lambda \in \varrho(A^*) = \varrho(A)$. Therefore, if $\omega(A) < \nu < \sigma < \pi$, then for functions $f \in H^1(\Sigma_\sigma)$ the operators $f(A^*)$ defined by Dunford calculus,

$$f(A^*) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z)R(z, A^*) dz,$$

are well defined and satisfy $f(A^*) = f(A)^*$.

Lemma 10.2.19. *Let A be a densely defined sectorial operator and let $\omega(A) < \sigma < \pi$. Then for all $x \in \overline{D(A) \cap R(A)} = \overline{R(A)}$ and $x^* \in \overline{D(A^*) \cap R(A^*)}$, we have*

$$\begin{aligned} \frac{1}{2M_{A,\sigma}} \|x\| &\leq \sup \{ |\langle x, y^* \rangle| : y^* \in D(A^*) \cap R(A^*), \|y^*\| \leq 1 \} \leq \|x\| \\ \frac{1}{2M_{A,\sigma}} \|x^*\| &\leq \sup \{ |\langle y, x^* \rangle| : y \in D(A) \cap R(A), \|y\| \leq 1 \} \leq \|x^*\|. \end{aligned}$$

Proof. It suffices to prove the left-hand estimates. Fix $x \in \overline{D(A) \cap R(A)}$ and put

$$\|x\| := \sup \{ |\langle x, y^* \rangle| : y^* \in D(A^*) \cap R(A^*), \|y^*\| \leq 1 \}.$$

Choose $x^* \in X^*$ such that $|\langle x, x^* \rangle| = \|x\|$. As in Proposition 10.2.6 we have $\zeta_n(A)x \rightarrow x$ as $n \rightarrow \infty$, $y_n^* := \zeta_n(A^*)x^* \in D(A^*) \cap R(A^*)$, and $\|y_n^*\| \leq 2M_{\sigma,A}\|x^*\|$. Then

$$\|x\| = |\langle x, x^* \rangle| = \lim_{n \rightarrow \infty} |\langle \zeta_n(A)x, x^* \rangle| = \lim_{n \rightarrow \infty} |\langle x, y_n^* \rangle| \leq 2M_{\sigma,A}\|x^*\|.$$

This proves the first assertion. The proof of the second is similar. $\square$

Proposition 10.2.20 (Adjoint). *Let A be a densely defined sectorial operator with a bounded H^∞ -calculus. Then A^* has a bounded H^∞ -calculus, we have $\omega_{H^\infty}(A) = \omega_{H^\infty}(A^*)$, and for all $f \in H^\infty(\Sigma_\sigma)$ with $\omega_{H^\infty}(A) < \sigma < \pi$ we have*

$$\langle x, f(A^*)x^* \rangle = \langle f(A)x, x^* \rangle, \quad x \in \overline{D(A) \cap R(A)}, \quad x^* \in \overline{D(A^*) \cap R(A^*)}.$$

Proof. For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, the identity $f(A)^* = f(A^*)$ implies $\langle x, f(A^*)x^* \rangle = \langle f(A)x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$ and therefore

$$|\langle x, f(A^*)x^* \rangle| \leq M_{\sigma,A}^\infty \|f\|_{H^\infty(\Sigma_\sigma)} \|x\| \|x^*\|.$$

Taking the supremum over all $x \in X$ with $\|x\| \leq 1$, we see that A^* has a bounded $H^\infty(\Sigma_\sigma)$ -calculus with angle $\omega_{H^\infty}(A^*) \leq \omega_{H^\infty}(A)$.

Now let $f \in H^\infty(\Sigma_\sigma)$ and let ζ_n be as in Proposition 10.2.6. Then by the convergence property of Theorem 10.2.13, for all $x \in \overline{D(A) \cap R(A)}$ and $x^* \in \overline{D(A^*) \cap R(A^*)}$ we obtain

$$\langle x, f(A^*)x^* \rangle = \lim_{n \rightarrow \infty} \langle x, (\zeta_n f)(A^*)x^* \rangle = \lim_{n \rightarrow \infty} \langle (\zeta_n f)(A)x, x^* \rangle = \langle f(A)x, x^* \rangle,$$

using that $\zeta_n f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ in the middle step, so that the duality identity of the first part of the proof applies.

To finish the proof it remains to be shown that the converse inequality $\omega_{H^\infty}(A) \leq \omega_{H^\infty}(A^*)$ holds. But this follows by applying what has been said above to the part of A^* in $\overline{D(A^*) \cap R(A^*)}$ and using the lemma to embed X isomorphically onto a closed subspace of the dual of $\overline{D(A^*) \cap R(A^*)}$. $\square$

10.2.c First examples

We will now present some first examples of operators having a bounded H^∞ -calculus. They only serve to give a first flavour of the subject, and most of them admit far-reaching generalisations, some of which are collected in the Notes at the end of the chapter.

Proposition 10.2.21 (Bounded operators). *Let A be a bounded operator whose spectrum $\sigma(A)$ avoids the negative axis $(-\infty, 0]$. For any $0 < \sigma < \pi$ such that $\sigma(A) \subseteq \Sigma_\sigma$, A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus.*

Proof. We define the calculus by the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A) \, dz,$$

where Γ is a simple closed Jordan curve contained in Σ_σ containing the spectrum of A in its interior, oriented in the counter-clockwise direction. If $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, then by Cauchy's theorem the contour Γ may be changed to $\partial\Sigma_\sigma$, and therefore the above definition agrees with the Dunford calculus of A as a sectorial operator. $\square$

The condition $0 \notin \sigma(A) \cap (-\infty, 0]$ cannot be omitted, as can be seen from Corollary 10.2.29.

Proposition 10.2.22 (Pointwise multipliers). *Let $(S, \mathcal{A}, \mu)$ be a measure space, let $0 < \sigma < \pi$, and let $m : S \rightarrow \mathbb{C}$ be a measurable function taking values in Σ_σ μ -almost surely. Let $1 \leq p < \infty$ and consider the pointwise multiplication operator A_m on $L^p(S)$ given by*

$$M_m \phi(s) := m(s) \phi(s), \quad s \in S,$$

with its natural domain $\mathcal{D}(M_m) = \{\phi \in L^p(S) : m\phi \in L^p(S)\}$. Then A_m is a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus on $L^p(S)$.

Proof. For $z \notin \overline{\Sigma_\sigma}$ we have $z \in \varrho(A_m)$ and $R(z, A_m)\phi(s) = (z - m(s))^{-1} f(s)$. Fix $\omega < \nu < \sigma$. For functions $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $\phi \in L^p(S)$ we then have

$$f(A_m)\phi(s) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \frac{f(z)}{z - m(s)} \phi(s) \, dz = f(m(s))\phi(s)$$

for μ -almost all $s \in S$. Clearly, $\|f(A_m)\| \leq \|f\|_\infty$ and therefore A_m has a bounded $H^\infty(\Sigma_\sigma)$ -calculus on $L^p(S)$, with constant $M_{\sigma, A_m}^\infty \leq 1$. $\square$

We continue with two basic Hilbertian examples.

Proposition 10.2.23 (Positive self-adjoint operators). *Let A be a densely defined self-adjoint operator on a complex Hilbert space H which is positive in the sense that the spectrum $\sigma(A)$ is contained in the positive half-line $[0, \infty)$. Then A has a bounded H^∞ -calculus of angle 0.*

Proof. By Proposition 10.2.11 it suffices to consider the part of A in $H_0 := \overline{D(A) \cap R(A)}$. Considered as an operator on that space, A is positive, self-adjoint and injective by Proposition 10.1.8.

By the spectral theorem for self-adjoint operators, A is unitarily equivalent to a multiplication operator. More precisely, there exists a measure space $(S, \mathcal{A}, \mu)$ and measurable function $m : S \rightarrow [0, \infty)$ and an invertible operator $U \in \mathcal{L}(H_0, L^2(S))$ such that

$$A = U^{-1} A_m U,$$

where A_m is as in Proposition 10.2.22. Since A is injective, A_m is injective and hence $\{s \in S : m(s) = 0\}$ has measure zero. Since the property of having a bounded H^∞ -calculus is inherited under similarity transforms, from Proposition 10.2.22 we conclude that A is sectorial and has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) = 0$. $\square$

Theorem 10.2.24 (Contraction semigroups on Hilbert spaces). *Suppose that $-A$ generates a C_0 -contraction semigroup on a Hilbert space H . Then A has a bounded $H^\infty(\Sigma_\sigma)$ -functional calculus for all $\frac{1}{2}\pi < \sigma < \pi$ and*

$$\|f(A)\| \leq \|f\|_{H^\infty(\Sigma_{\frac{1}{2}\pi})}, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma).$$

In particular, this calculus has angle $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$ and constants $M_{\sigma,A}^\infty \leq 1$ for all $\frac{1}{2}\pi < \sigma < \pi$.

In Theorem 10.4.21 it will be shown that for any sectorial operator A with a bounded H^∞ -calculus on a Hilbert space one has equality $\omega_{H^\infty}(A) = \omega(A)$. Moreover, a characterisation of those operators satisfying $\omega_{H^\infty}(A) < \frac{1}{2}\pi$ will be given in Theorem 10.4.22.

Proof. We prove the theorem in three steps.

Step 1 – First we assume in addition that A is a *bounded* operator and that $-A + c$ generates a C_0 -contraction semigroup on H for some fixed $c > 0$, or equivalently (see Corollary G.4.5) that $\Re(Ax|x) \geq c\|x\|^2$ for all $x \in H$. Observe that $\{\lambda \in \mathbb{C} : \Re(\lambda) > -c\} \subseteq \rho(A)$ and since $\sigma(A)$ is compact, A is sectorial of angle $\frac{1}{2}\pi - \delta$ for some $\delta > 0$. Next note that

$$((A + A^*)x|x) = (Ax|x) + \overline{(Ax|x)} = 2\Re(Ax|x) \geq 2c\|x\|^2,$$

so that $A + A^*$ is self-adjoint and positive.

Fix $\frac{1}{2}\pi < \sigma < \pi$ and let $f \in H^1(\Sigma_\sigma)$. Then $\lambda \mapsto f(\lambda)(A^* + \lambda)^{-1}$ is holomorphic in $\overline{\mathbb{C}\Sigma_{\frac{1}{2}\pi - \delta}}$. By Cauchy's theorem applied to the boundary of the semi-annulus $\{r \leq |z| \leq R\} \cap \{\Re z \geq 0\}$, upon letting $r \downarrow 0$ and $R \rightarrow \infty$ and using Lemma H.1.4, we obtain

$$\int_{i\mathbb{R}} f(\lambda)(A^* + \lambda)^{-1} d\lambda = 0.$$

Hence,

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \int_{i\mathbb{R}} f(\lambda) R(\lambda, A) d\lambda \\ &= -\frac{1}{2\pi i} \int_{i\mathbb{R}} f(\lambda) [(A - \lambda)^{-1} + (A^* + \lambda)^{-1}] d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f(it) (A^* + it)^{-1} (A^* + A) (A - it)^{-1} dt. \end{aligned}$$

Therefore, for all $x, y \in H$,

$$\begin{aligned} (f(A)x|y) &= \frac{1}{2\pi} \int_{\mathbb{R}} f(it) \left((A^* + A)^{1/2} (A - it)^{-1} x \middle| (A^* + A)^{1/2} (A - it)^{-1} y \right) dt. \end{aligned} \quad (10.17)$$

So far we took $f \in H^1(\Sigma_\sigma)$. Taking $f = \zeta_n$ as in Proposition 10.2.6 and letting $n \rightarrow \infty$, and taking $x = y$ in (10.17), for all $x \in H$ (remember that in this step A is bounded and invertible) we obtain

$$\|x\|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \|(A^* + A)^{1/2} (A - it)^{-1} x\|^2 dt. \quad (10.18)$$

It now follows from (10.17) and (10.18) that

$$\begin{aligned} |(f(A)x, y)| &\leq \|f\|_{H^\infty(\Sigma_{\frac{1}{2}\pi})} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|(A^* + A)^{1/2} (A - it)^{-1} x\|^2 dt \right)^{1/2} \\ &\quad \times \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|(A^* + A)^{1/2} (A - it)^{-1} y\|^2 dt \right)^{1/2} \\ &= \|f\|_{H^\infty(\Sigma_{\frac{1}{2}\pi})} \|x\| \|y\|. \end{aligned}$$

Step 2 – We reduce the general case to the case considered in Step 1. The assumptions of the theorem imply that A is sectorial of angle $\frac{1}{2}\pi$.

For any $\varepsilon > 0$ define a bounded operator

$$A_\varepsilon := ((A + \varepsilon)^{-1} + \varepsilon)^{-1} = \varepsilon^{-1} - \varepsilon^{-2}(\varepsilon + \varepsilon^{-1} + A)^{-1}.$$

Then $\|(\varepsilon + \varepsilon^{-1} + A)^{-1}\| \leq (\varepsilon + \varepsilon^{-1})^{-1} = \varepsilon(\varepsilon^2 + 1)^{-1}$ and therefore

$$\|e^{-tA_\varepsilon}\| \leq e^{-\varepsilon^{-1}t} e^{\varepsilon^{-2}t\|(\varepsilon + \varepsilon^{-1} + A)^{-1}\|} \leq e^{-\varepsilon(1+\varepsilon^2)^{-1}t}.$$

Hence A_ε satisfies the assumption of Step 1 and in particular A_ε is sectorial of angle $\frac{1}{2}\pi$ with uniform estimates in ε . Moreover, for all $\Re \lambda < 0$,

$$\begin{aligned} \|R(\lambda, A) - R(\lambda, A_\varepsilon)x\| &= \|R(\lambda, A_\varepsilon)(A - A_\varepsilon)R(\lambda, A)x\| \\ &\leq \frac{1}{\Re(\lambda)} \|(A - A_\varepsilon)R(\lambda, A)x\|. \end{aligned}$$

The right-hand side tends to zero as $\varepsilon \downarrow 0$ since for all $x \in \mathcal{D}(A)$,

$$\|Ax - A_\varepsilon x\| \leq \|Ax - \varepsilon^{-1}(\varepsilon + \varepsilon^{-1} + A)^{-1}Ax\| + \|(\varepsilon + \varepsilon^{-1} + A)^{-1}x\| \rightarrow 0$$

by Proposition 10.1.7 and the sectoriality of A . Therefore, for $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $\frac{1}{2}\pi < \nu < \sigma$, by dominated convergence we obtain

$$\begin{aligned} f(A)x &= \int_{\partial\Sigma_\omega} f(\lambda)R(\lambda, A)x \, d\lambda \\ &= \lim_{\varepsilon \downarrow 0} \int_{\partial\Sigma_\omega} f(\lambda)R(\lambda, A_\varepsilon)x \, d\lambda \\ &= \lim_{\varepsilon \downarrow 0} \int_{i\mathbb{R}} f(\lambda)R(\lambda, A_\varepsilon)x \, d\lambda = \lim_{\varepsilon \downarrow 0} f(A_\varepsilon)x. \end{aligned}$$

By Step 1,

$$\|f(A_\varepsilon)x\| \leq \|f\|_{H^\infty(\Sigma_{\frac{1}{2}\pi})}\|x\|, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma),$$

and the estimate $\|f(A)x\| \leq \|f\|_{H^\infty(\Sigma_{\frac{1}{2}\pi})}\|x\|$ follows by letting $\varepsilon \downarrow 0$. This completes the proof. $\square$

In the setting of general Banach spaces, an essential feature in the proof of Theorem 10.2.24 persists, namely the norm equivalence given in (10.18) in terms of A . It will be shown in Section 10.4.a that, modulo some assumptions that are automatically satisfied in the Hilbertian case, such norm equivalences resurface as square function estimates and characterise sectorial operators with a bounded H^∞ -calculus.

Let us also point out that an analogue of Theorem 10.2.24 holds for positive C_0 -contraction semigroups on L^p -spaces when $1 < p < \infty$ (see Theorem 10.7.12).

Recall from Example 10.1.5 that the operator $A = -\Delta$ is densely defined, injective and sectorial of angle $\omega(A) = 0$.

Theorem 10.2.25 (Laplacian on $L^p(\mathbb{R}^d; X)$). *Let X be a UMD Banach space and $1 < p < \infty$. The Laplace operator $-\Delta$ has a bounded H^∞ -calculus on $L^p(\mathbb{R}^d; X)$ of angle 0.*

The UMD assumption is necessary, as a consequence of Corollary 10.5.2 below.

Proof. Define, for $f \in H^\infty(\Sigma_\sigma)$ with $0 < \sigma < \pi$,

$$\Psi(f)\phi := \mathcal{F}^{-1}[f(4\pi^2|\cdot|^2)\widehat{\phi}(\cdot)], \quad \phi \in \mathcal{S}(\mathbb{R}^d; X).$$

To prove the boundedness of $f(A)$ we check the conditions of the Mihlin multiplier theorem (Theorem 5.5.10). For this we have to bound, for multi-indices $\alpha \leq (1, \dots, 1)$,

$$|t|^{|\alpha|} D^\alpha f(|t|^2) = |t|^{2|\alpha|} 2^{|\alpha|} t^\alpha f^{(|\alpha|)}(|t|^2) = \left(\frac{2t}{|t|}\right)^\alpha (|t|^2)^{|\alpha|} f^{(|\alpha|)}(|t|^2).$$

But for integers $k \geq 1$, the boundedness of $t^k f^{(k)}(t)$ for $t > 0$ follows from the Cauchy formula since $f \in H^\infty(\Sigma_\sigma)$ for $\sigma > 0$. Indeed, for $\sigma' \in (0, \sigma)$, we have

$$|t^k f^{(k)}(t)| \leq \frac{1}{2\pi} \int_{\Gamma_{\sigma'}} \frac{|t|^k |f(z)|}{|z - t|^{k+1}} |dz| \leq \frac{C}{2\pi} \int_{\Gamma_{\sigma'}} \frac{1}{|z - 1|^{k+1}} |dz| < \infty.$$

Hence $\Psi(f)$ defines a bounded operator on $L^p(\mathbb{R}^d; X)$. One can check directly that the mapping $f \mapsto \Psi(f)$ satisfies the assumptions of Theorem 10.2.14. For instance, the multiplicativity follows from the properties of the convolution product, and the convergence property follows from the dominated convergence theorem since $\hat{\phi} \in L^1(\mathbb{R}^d; X)$ and $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $L^p(\mathbb{R}^d; X)$. By Theorem 10.2.14, $f \mapsto \Psi(f)$ must be the bounded $H^\infty(\Sigma_\sigma)$ -calculus of $-\Delta$. Since $\sigma > 0$ was arbitrary, the result follows. $\square$

In the same way one proves that the first derivative $Af = f'$ on $L^p(\mathbb{R}; X)$ is sectorial and has a bounded $H^\infty(\Sigma_\omega)$ -calculus with $\omega(A) = \omega_{H^\infty}(A) = \frac{1}{2}\pi$ if $1 < p < \infty$ and X is a UMD space. Indeed, for $f \in H^\infty(\Sigma_\sigma)$ with $\sigma \in (\frac{1}{2}\pi, \pi)$ define

$$\Psi(f)\phi := \mathcal{F}^{-1}[f(2\pi i \cdot) \hat{\phi}(\cdot)], \quad \phi \in \mathcal{S}(\mathbb{R}^d; X).$$

The boundedness of $\Psi(A)$ again follows from the Mihlin multiplier theorem. In the same way as in the previous example one checks that $f \mapsto \Psi(f)$ defines a bounded $H^\infty(\Sigma_\sigma)$ -calculus for A .

Remark 10.2.26. Let $\sigma \in (0, \pi)$ and $1 < p < \infty$. The UMD property is not only sufficient for the boundedness of the $H^\infty(\Sigma_\sigma)$ functional calculus of $-\Delta$ on $L^p(\mathbb{R}^d; X)$, but also necessary. As a consequence the UMD property of X can be characterised in terms of the spectral theory of the operator $-\Delta$ on $L^p(\mathbb{R}^d; X)$. This point will be taken up in Section 10.5.

Counterexamples

In this section we discuss a class of diagonal operators modelled on a Schauder basis. It will be convenient later on to label the bases with index set $\mathbb{Z}$. From Chapter 4 we recall that a sequence $(x_n)_{n \in \mathbb{Z}}$ in a Banach space X is called a *Schauder basis* if for each $x \in X$ there exists a unique sequence of scalars $(\alpha_n)_{n \in \mathbb{Z}}$ such that $x = \sum_{n \in \mathbb{Z}} \alpha_n x_n$, the summation being understood in the improper sense, i.e., $x = \lim_{m, n \rightarrow \infty} \sum_{k=-m}^n \alpha_k x_k$ (see Definition 4.1.14).

As a side-remark on double-sided improper convergence, observe that the limit $\lim_{m, n \rightarrow \infty} \sum_{k=-m}^n \alpha_k x_k$ exists if and only if both $\lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k x_k$ and $\lim_{m \rightarrow \infty} \sum_{k=-m}^0 \alpha_k x_k$ exist. The ‘if’ part is clear. To prove the ‘only if’ part, let $x = \lim_{m, n \rightarrow \infty} \sum_{k=-m}^n \alpha_k x_k$ be the limit. Given an $\varepsilon > 0$ we choose N so large that

$$\left\| x - \sum_{k=-m}^n \alpha_k x_k \right\| < \varepsilon$$

for all $m, n \geq N$. Then, for $m, n \geq N$,

$$\left\| \sum_{k=1}^n \alpha_k x_k - \sum_{k=1}^m \alpha_k x_k \right\| = \left\| \sum_{k=-N}^n \alpha_k x_k - \sum_{k=-N}^m \alpha_k x_k \right\| < 2\varepsilon.$$

Therefore the series $\sum_{k=1}^\infty \alpha_k x_k$ converges. In the same way one sees that $\sum_{k=-\infty}^0 \alpha_k x_k$ converges.

Now let $(x_n)_{n \in \mathbb{Z}}$ be a Schauder basis. By an application of the closed graph theorem, the coordinate projections

$$x_j^* : \sum_{n \in \mathbb{Z}} \alpha_n x_n \mapsto \alpha_j$$

are bounded. Therefore, for $M, N \in \mathbb{Z}$ with $M \leq N$ we can define the projections $P_{M,N} \in \mathcal{L}(X)$ by

$$P_{M,N} : \sum_{n \in \mathbb{Z}} \alpha_n x_n \mapsto \sum_{n=M}^N \alpha_n x_n.$$

By the uniform boundedness principle, we have

$$C := \sup_{M \leq N} \|P_{M,N}\| < \infty.$$

Throughout this subsection the constant C will have the above meaning.

We start with a multiplier lemma of Marcinkiewicz type.

Lemma 10.2.27. *Let X be a Banach space with a Schauder basis $(x_n)_{n \in \mathbb{Z}}$. Assume that $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of scalars such that*

$$L := \sup_{n \in \mathbb{Z}} |\lambda_n| + \sum_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| < \infty.$$

If $x = \sum_{n \in \mathbb{Z}} c_n x_n$, then $\sum_{n \in \mathbb{Z}} \lambda_n c_n x_n$ converges in X and

$$\left\| \sum_{n \in \mathbb{Z}} \lambda_n c_n x_n \right\| \leq 2CL \|x\|.$$

Proof. As explained before the lemma it suffices to consider bases over the index sets $\mathbb{Z}_+$ and $\mathbb{Z}_-$ separately. By relabelling we may assume that $c_j = 0$ for $j \leq 0$. Let $s_n = \sum_{j=1}^n c_j x_j$ and $t_n = \sum_{j=1}^n \lambda_j c_j x_j$ for $n \geq 1$. A summation by parts gives

$$t_n = \lambda_{n+1} s_n - \sum_{j=1}^n (\lambda_{j+1} - \lambda_j) s_j.$$

The boundedness of the partial sum projections implies $\|s_j\| \leq C\|x\|$ and thus

$$\|t_n\| \leq |\lambda_{n+1}|\|s_n\| + \sum_{j=1}^n |\lambda_{j+1} - \lambda_j|\|s_j\| \leq CL\|x\|.$$

Therefore, the required estimate follows as soon as we have shown that $(t_n)_{n \geq 1}$ is convergent. Applying the above estimate with $s_n - s_m$ (where $1 \leq m \leq n$) instead of x (and thus $c_1 = \dots = c_m = 0$), we obtain $\|t_n - t_m\| \leq CL\|s_n - s_m\|$. It follows that $(t_n)_{n \geq 1}$ is a Cauchy sequence, hence convergent. $\square$

A Schauder basis $(x_n)_{n \in \mathbb{Z}}$ is called *unconditional* if there exists a constant $M \geq 0$ such that for all finitely non-zero sequences $(c_n)_{n \in \mathbb{Z}}$ and scalar sequences of modulus one $(\epsilon_n)_{n \in \mathbb{Z}}$ we have

$$\left\| \sum_{n \in \mathbb{Z}} \epsilon_n c_n x_n \right\| \leq M \left\| \sum_{n \in \mathbb{Z}} c_n x_n \right\|.$$

A Schauder basis $(x_n)_{n \in \mathbb{Z}}$ is called *conditional* if it is not unconditional.

Proposition 10.2.28. *On a Banach space X with a Schauder basis $(x_n)_{n \in \mathbb{Z}}$ consider the diagonal operator A defined by*

$$A \sum_{n \in \mathbb{Z}} c_n x_n = \sum_{n \in \mathbb{Z}} 2^n c_n x_n, \\ \mathcal{D}(A) = \left\{ \sum_{n \in \mathbb{Z}} c_n x_n \in X : \sum_{n \in \mathbb{Z}} 2^n c_n x_n \text{ converges in } X \right\}.$$

Then:

- (1) A is sectorial with $\omega(A) = 0$, and A is densely defined, injective, and has dense range.
- (2) A has a bounded H^∞ -calculus if and only if $(x_n)_{n \in \mathbb{Z}}$ is unconditional.

As a consequence of (1) and Theorem G.5.2, $-A$ generates an analytic C_0 -semigroup which is uniformly bounded and strongly continuous on Σ_ω for every $\omega \in (0, \frac{1}{2}\pi)$. This can also be shown directly by a variation of the proof below.

Proof. In order to show that A is sectorial with $\omega(A) = 0$, let $\omega \in (0, \frac{1}{2}\pi)$. We will frequently use the following estimate (see Lemma 10.2.4):

$$|1 + z| \geq (|z| \vee 1) \sin(\omega), \quad t > 0, \quad z \in \Sigma_{\pi-\omega}. \quad (10.19)$$

Let $z \in \Sigma_{\pi-\omega}$. Then with $\lambda_n(z) = (z + 2^n)^{-1}$, $n \in \mathbb{Z}$, we have

$$\phi_n(z) := |z| |\lambda_n(z)| = \frac{2^{-n} |z|}{|2^{-n} z + 1|} \leq 1 + \frac{1}{|2^{-n} z + 1|} \leq \frac{2}{\sin(\omega)},$$

where we used (10.19) in the last step. Similarly, let

$$\psi(z) := |z| \sum_{n \in \mathbb{Z}} |\lambda_{n+1}(z) - \lambda_n(z)| = \sum_{n \in \mathbb{Z}} \frac{|z| 2^{-(n+1)}}{|2^{-(n+1)}z + 1| |2^{-n}z + 1|}.$$

Let $m \in \mathbb{Z}$ be the unique integer such that $2^m < |z| \leq 2^{m+1}$. For $n \leq m-1$, by (10.19) we can estimate

$$\frac{|z| 2^{-(n+1)}}{|2^{-(n+1)}z + 1| |2^{-n}z + 1|} \leq \frac{2}{\sin(\omega)} \frac{1}{|2^{-n}z + 1|} \leq \frac{2}{\sin^2(\omega)} 2^{-m+n}.$$

For $n \geq m$ we have

$$\frac{|z| 2^{-(n+1)}}{|2^{-(n+1)}z + 1| |2^{-n}z + 1|} \leq \frac{2^{m-n-1}}{\sin^2(\omega)}.$$

Therefore,

$$\psi(z) \leq \frac{2}{\sin^2(\omega)} \sum_{n \leq m-1} 2^{-m+n} + \frac{1}{\sin^2(\omega)} \sum_{n \geq m} 2^{m-n-1} = \frac{3}{\sin^2(\omega)}.$$

By Lemma 10.2.27, the mapping $R(z)(\sum_{n \in \mathbb{Z}} \alpha_n x_n) := \sum_{n \in \mathbb{Z}} \lambda_n(z) \alpha_n x_n$ is bounded on X and

$$|z| \|R(z)\| \leq 2C(\phi(z) + \psi(z)) \leq \frac{4C}{\sin(\omega)} + \frac{6C}{\sin^2(\omega)}.$$

It follows that $z \in \varrho(-A)$ and $R(z, -A) = R(z)$. This shows that $\Sigma_{\pi-\omega} \subseteq \varrho(A)$, so $\mathbb{C}\overline{\Sigma_\omega} \subseteq \varrho(A)$, and the above bound shows that A is sectorial of angle $\omega(A) \leq \omega$. Injectivity of A is clear from the fact that $(x_n)_{n \in \mathbb{Z}}$ is a Schauder basis. Since $x_n \in D(A) \cap R(A)$, the density assertions follow as well.

Next we prove that the boundedness of the H^∞ -calculus of A is equivalent to the unconditionality of the basis $(x_n)_{n \in \mathbb{Z}}$. Suppose first that $(x_n)_{n \in \mathbb{Z}}$ is unconditional, say with unconditionality constant M . Then A admits a bounded H^∞ -calculus of zero angle, given by

$$f(A)x_n = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \frac{f(z)}{z - 2^n} x_n \, dz = f(2^n)x_n,$$

for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $0 < \nu < \sigma < \pi$. Clearly, the unconditionality implies $\|f(A)\| \leq M\|f\|_\infty$.

Conversely, suppose that the operator A has a bounded $H^\infty(\Sigma_\omega)$ -calculus for some $\omega \in (0, \pi)$. Let $\omega < \nu < \sigma < \pi$. Letting $\zeta(z) = z(1+z)^{-2}$ we have $\zeta(A)x_n = \zeta(2^n)x_n$. It follows that for $f \in H^\infty(\Sigma_\sigma)$, $f\zeta \in H^1(\Sigma_\sigma)$ and hence

$$f(A)x_n = \frac{1}{\zeta(2^n)} (f\zeta)(A)x_n$$

$$\begin{aligned}
&= \frac{1}{2\pi i \zeta(2^n)} \int_{\partial \Sigma_\nu} f(z) \zeta(z) R(z, A) x_n \, dz \\
&= \frac{1}{2\pi i \zeta(2^n)} \int_{\partial \Sigma_\nu} \frac{f(z) \zeta(z)}{z - 2^n} x_n \, dz = f(2^n) x_n.
\end{aligned}$$

By a classical interpolation theorem of complex analysis (see Proposition [H.2.7](#)), there exists a constant C_σ such that for any bounded sequence $(\alpha_n)_{n \in \mathbb{Z}}$ contained in Σ_σ there exists a function $f \in H^\infty(\Sigma_\sigma)$ of norm $\|f\|_\infty \leq C_\sigma$ such that $f(2^n) = \alpha_n$ for all $n \in \mathbb{Z}$. Then, for any finitely non-zero sequence $(c_n)_{n \in \mathbb{Z}}$,

$$\left\| \sum_{n \in \mathbb{Z}} \alpha_n c_n x_n \right\| = \left\| \sum_{n \in \mathbb{Z}} f(2^n) c_n x_n \right\| = \left\| f(A) \sum_{n \in \mathbb{Z}} c_n x_n \right\| \leq \|f(A)\| \left\| \sum_{n \in \mathbb{Z}} c_n x_n \right\|,$$

where we applied the identity $f(A)x_n = f(2^n)x_n$. Since $\|f(A)\| \leq C_\sigma M_{\omega, A}^\infty$, this proves that $(x_n)_{n \in \mathbb{Z}}$ is unconditional, with unconditionality constant at most $C_\sigma M_{\omega, A}^\infty$. $\square$

We leave it to the reader to check that Proposition [10.2.28](#) remains true if the index set I is replaced by either $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{<0}$. The resulting diagonal operators A then have some additional structure: in the case of $I = \mathbb{Z}_{>0}$, A is invertible, and in the case $I = \mathbb{Z}_{<0}$, A is a bounded operator by Lemma [10.2.27](#). An immediate consequence is the following result.

Corollary 10.2.29. *If X has a conditional Schauder basis $(x_n)_{n \geq 1}$, then the following hold:*

- (1) *there exists an invertible sectorial operator A with $\omega(A) = 0$ such that A does not have a bounded H^∞ -calculus.*
- (2) *there exists a bounded sectorial operator A with $\omega(A) = 0$ such that A does not have a bounded H^∞ -calculus.*

It can be shown that every Banach space with a Schauder basis has a conditional Schauder basis (see the Notes), but it would take us too far afield to prove this result. Instead we choose to present some more concrete examples below and refer the reader to the Notes for a further discussion.

Example 10.2.30. The standard trigonometric system $(e_n)_{n \in \mathbb{Z}}$, $e_n(x) = e^{2\pi i n x}$, is a Schauder basis in $L^p(\mathbb{T})$ for $1 < p < \infty$ by Proposition [5.2.7](#). Consider the diagonal operator

$$A\phi := \sum_{n \in \mathbb{Z}} 2^n \widehat{\phi}(n) e_n, \quad \phi \in L^p(\mathbb{T}),$$

with its natural domain consisting of all $\phi \in L^p(\mathbb{T})$ for which this series converges in $L^p(\mathbb{T})$. By Proposition [10.2.28](#), A is sectorial of angle 0. By Proposition [10.2.28](#), A has a bounded H^∞ -calculus if and only if $(e_n)_{n \in \mathbb{Z}}$ is an unconditional basis, and in Example [4.1.12](#) it was shown that this is the case if and only if $p = 2$.

In order to give an example of a sectorial operator on a Hilbert space without a bounded H^∞ -calculus we will use a variation of Example 10.2.30 on weighted spaces. The only fact which will be needed is that the periodic Hilbert transform $\tilde{H}$ introduced in Chapter 5,

$$\tilde{H}\left(\sum_{k \in \mathbb{Z}} a_k e_k\right) := -i \sum_{k \in \mathbb{Z}} \operatorname{sgn}(k) a_k e_k$$

is bounded on the weighted space $L^2(\mathbb{T}, w)$ with $w(x) = |x|^\alpha$ and $\alpha \in (-1, 1)$. In order to be self-contained we include a proof, which actually extends without additional difficulty to $p \in (1, \infty)$ and the UMD-valued setting.

Proposition 10.2.31. *Let X be a UMD Banach space and let $p \in (1, \infty)$. Let $\alpha \in (-1, p-1)$ and set $w(x) = |x|^\alpha$. Then the Hilbert transform H and the periodic Hilbert transform $\tilde{H}$ are bounded on $L^p(\mathbb{R}, w; X)$ and $L^p(\mathbb{T}, w; X)$, respectively.*

Proof. By density it suffices to prove that there exists a constant $C \geq 0$ such that $\|Hf\|_{L^p(\mathbb{R}, w; X)} \leq C\|f\|_{L^p(\mathbb{R}, w; X)}$ for all $f \in C_c(\mathbb{R} \setminus \{0\}; X)$. We start from the identity

$$Hf = w^{-1/p} H(w^{1/p} f) + Tf,$$

where

$$Tf(x) = \frac{1}{\pi} w(x)^{-1/p} \int_{\mathbb{R}} \frac{w(x)^{1/p} - w(y)^{1/p}}{x - y} f(y) dy.$$

Note that the right-hand side is well defined as a Bochner integral in X . The boundedness of the Hilbert transform in $L^p(\mathbb{R}; X)$, which was established in Theorem 5.1.13, gives

$$\|w^{-1/p} H(w^{1/p} f)\|_{L^p(\mathbb{R}, w; X)} = \|H(w^{1/p} f)\|_p \leq \|H\| \|w^{1/p} f\|_p = \|H\| \|f\|_{w, p}.$$

Accordingly it suffices to prove that $\|Tf\|_{L^p(\mathbb{R}, w; X)} \leq C\|f\|_{L^p(\mathbb{R}, w; X)}$. By the identity

$$|x|^{1/p} w(x)^{1/p} |Tf(x)| = \frac{1}{\pi} \int_{\mathbb{R}} k(x/y) |y|^{1/p} w(y)^{1/p} f(y) \frac{dy}{|y|},$$

where $k(y) = |y|^{1/p} \left| \frac{|y|^{\alpha/p} - 1}{y-1} \right|$, the latter integral can be viewed as a convolution on the multiplicative group $(\mathbb{R} \setminus \{0\}, \cdot)$ with Haar measure $\frac{dy}{|y|}$. Using the assumption $\alpha \in (-1, p-1)$ with $p \in (1, \infty)$, it is easy to check that $k \in L^1(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})$. Therefore, by Young's inequality,

$$\begin{aligned} \|Tf\|_{L^p(\mathbb{R}, w; X)} &= \left\| \left| \cdot \right|^{1/p} w^{1/p} Tf \right\|_{L^p(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})} \\ &\leq \|k\|_{L^1(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})} \left\| \left| \cdot \right|^{1/p} w^{1/p} f \right\|_{L^p(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})} \end{aligned}$$

$$= \|k\|_{L^1(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})} \|f\|_{L^p(\mathbb{R}, w; X)}.$$

This gives the desired estimate, with $C = \|k\|_{L^1(\mathbb{R} \setminus \{0\}, \frac{dy}{|y|})}$.

The boundedness of $\tilde{H}$ follows from the boundedness of H by repeating the transference argument of Proposition 5.2.5, replacing dx by $w(x) dx$ at the appropriate places. $\square$

Example 10.2.32. Let again $\alpha \in (-1, p-1)$ with $p \in (1, \infty)$ and $w(x) = |x|^\alpha$. Using Proposition 10.2.31, the proof of Proposition 5.2.7 extends to the weighted setting and we find that the trigonometric system $(e_n)_{n \in \mathbb{Z}}$ is a Schauder basis in $L^p(\mathbb{T}, w)$.

For $f \in L^p(\mathbb{T}, w)$ let

$$A\phi := \sum_{n \in \mathbb{Z}} 2^n \hat{\phi}(n) e_n$$

with its natural domain. By Proposition 10.2.28, A is sectorial of angle 0. We will show that A has a bounded H^∞ -calculus if and only if $p = 2$ and $\alpha = 0$.

It suffices to prove the ‘only if’ part. Let us assume that A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for some $\sigma \in (0, \pi)$. As in Example 10.2.30, an application of Proposition 10.2.28 gives the necessity of the condition $p = 2$. To show that $\alpha = 0$ we note that the operators A^{is} , $s \in \mathbb{R}$, are bounded on $L^p(\mathbb{T}, w)$ by the boundedness of the $H^\infty(\Sigma_\sigma)$ -calculus on this space (note that $z \mapsto z^{is} = \exp(is(\log|z| + i \arg(z)))$ belongs to $H^\infty(\Sigma_{\pi-\delta})$ for every $\delta \in (0, \pi)$). The following argument shows that the boundedness of operators A^{is} for some $s \notin \frac{1}{\log(2)}\mathbb{N}$ forces $\alpha = 0$.

As in the proof of Proposition 10.2.28, the H^∞ -calculus of A is given by $f(A)e_n = f(2^n)e_n$ for $f \in H^\infty(\Sigma_\sigma)$. In particular, taking $f(z) = z^{is}$, we find that $A^{is}e_n = e^{2\pi i \log(2)sn}e_n = T(s \log(2))e_n$, where $T(s)$ denotes right-translation over s . Thus we find that $T(s)$ is a bounded operator on $L^p(\mathbb{T}, w)$. But it is easy to check that this is only true for $\alpha = 0$.

Later we will see that this example admits a different interpretation: it provides an operator A on $L^p(\mathbb{T}, w)$ which has bounded imaginary powers for all $p \in (1, \infty)$ but has a bounded H^∞ -calculus only for $p = 2$.

10.3 R -boundedness of the H^∞ -calculus

This section is devoted to a study of the R -boundedness of various classes of operators of the form $f(A)$, where A is a sectorial operator, possibly with a bounded H^∞ -calculus. We remind the reader of the fact that throughout this chapter we work over the complex scalar field. ‘ R -boundedness’ therefore refers to random sums with complex Rademacher variables. At the expense of an additional numerical constant the corresponding results for real Rademacher sums hold as well and follow by estimating random sums involving real Rademachers in terms of the corresponding sums involving complex

Rademachers by means of Proposition 6.1.19. Most proofs, however, also work for real Rademachers, and in those cases no additional constant is obtained. We let the interested reader check the particular instances at hand.

10.3.a R -sectoriality

As a first step in our programme it is of interest to study the R -boundedness of the operators $zR(z, A)$ as they appear in the Dunford calculus by writing

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma} f(z)[zR(z, A)] \frac{dz}{z}.$$

This leads to the notion of R -sectoriality:

Definition 10.3.1 (R -sectorial operators). A sectorial operator A is called R -sectorial if for some $\omega(A) < \sigma < \pi$ the family

$$\mathcal{T}_{\sigma, A} := \{zR(z, A) : z \in \mathbb{C}\overline{\Sigma_\sigma}\}$$

is R -bounded.

The angle of R -sectoriality is the number

$$\omega_R(A) := \inf \{\sigma \in (\omega(A), \pi) : \mathcal{T}_{\sigma, A} \text{ is } R\text{-bounded}\}.$$

For $\omega_R(A) < \sigma < \pi$, the R -bound of $\mathcal{T}_{\sigma, A}$ will be denoted by $M_{\sigma, A}^R$.

The class of γ -sectorial operators is defined similarly, replacing R -boundedness by γ -boundedness. The associated angle and bounds are denoted by $\omega_\gamma(A)$ and $M_{\sigma, A}^\gamma$. By Theorem 8.1.3, R -boundedness implies γ -boundedness, and thus every R -sectorial operator is γ -sectorial, and when X has finite co-type these notions are equivalent.

The Dunford calculus of an R -sectorial operator is R -bounded:

Proposition 10.3.2 (R -boundedness of the Dunford calculus). Let A be an R -sectorial operator on X and let $\omega_R(A) < \sigma < \pi$. Then the set

$$\{f(A) : f \in H^1(\Sigma_\sigma), \|f\|_{H^1(\Sigma_\sigma)} \leq 1\}$$

is R -bounded, with R -bound at most $\frac{1}{\pi} M_{\sigma, A}^R$.

Proof. Fixing an arbitrary $\omega_R(A) < \nu < \sigma$, we have

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z)[zR(z, A)] \frac{dz}{z}.$$

Since A is R -sectorial, $\{zR(z, A) : z \in \partial\Sigma_\nu \setminus \{0\}\}$ is R -bounded with constant $M_{\nu, A}^R$. Hence, by Theorem 8.5.2, the set in the statement of the proposition is R -bounded, with R -bound at most $\frac{1}{\pi} M_{\nu, A}^R$. Since this is true for all $\omega_R(A) < \nu < \pi$, by Proposition 8.1.22 we may pass to the limit $\nu \uparrow \sigma$ to obtain the desired bound $\frac{1}{\pi} M_{\sigma, A}^R$. $\square$

From Theorem G.5.2 we know that if A is densely defined and sectorial of angle $\omega(A) < \frac{1}{2}\pi$, then $-A$ generates a bounded analytic C_0 -semigroup $(S(\zeta))_{\zeta \in \Sigma_\eta}$ for every $\eta \in (0, \frac{1}{2}\pi - \omega(A))$, given by an inverse Laplace transform representation. By the same proof as in Proposition 10.3.2, this representation implies that if A is R -sectorial and $\omega(A) < \frac{1}{2}\pi$, then $(S(\zeta))_{\zeta \in \Sigma_\eta}$ is R -bounded for every $\eta \in (0, \frac{1}{2}\pi - \omega_R(A))$. In fact more is true:

Proposition 10.3.3. *Let A be a densely defined sectorial operator of angle $< \frac{1}{2}\pi$ on a Banach space X and let $(S(t))_{t \geq 0}$ be the bounded analytic C_0 -semigroup generated by $-A$ (see Theorem G.5.2). The following assertions are equivalent:*

- (1) A is R -sectorial with $\omega_R(A) < \frac{1}{2}\pi$;
- (2) $\{tR(it, A) : t \in \mathbb{R} \setminus \{0\}\}$ is R -bounded;
- (3) $\{S(z) : z \in \Sigma_\omega\}$ is R -bounded for some $\omega \in (0, \frac{1}{2}\pi)$;
- (4) $\{S(t) : t \geq 0\}$ and $\{tAS(t) : t \geq 0\}$ are R -bounded.

In this situation we have $\sup\{\omega \in (0, \frac{1}{2}\pi) : (3) \text{ holds}\} = \frac{1}{2}\pi - \omega_R(A)$.

From the proof it is easy to extract explicit estimates for the R -bounds of the sets in (3) and (4) analogous to the one of Proposition 10.3.2. Likewise the proof produces explicit estimates for the abscissa $\omega_R(A)$ and the constants $M_{\sigma, A}^R$ in terms of the R -bounds in (3) and (4).

Proof. (1) $\Leftrightarrow$ (2): Repeating the proof of Proposition 10.1.6, the condition in (2) is seen to be equivalent to the R -boundedness of $\lambda R(\lambda, A)$ on two rotated sectors $\pm i\Sigma_\omega$ for some $0 < \omega < \frac{1}{2}\pi$. By Proposition 8.5.8 the latter implies the R -boundedness of $\lambda R(\lambda, A)$ in the remaining part of the left half-plane.

(1) $\Rightarrow$ (4): The R -boundedness of the set $\{S(t) : t > 0\}$ is an immediate consequence of Propositions 10.2.7; and 10.3.2; alternatively one can use the inverse Laplace representation formula for $S(t)$ and proceed as in the proof of Proposition 10.3.2. Note that both arguments also establish the implication (1) $\Rightarrow$ (3). The R -boundedness of the set $\{tAS(t) : t > 0\}$ follows from Proposition 10.3.2 in combination with Lemma 10.2.8.

(4) $\Rightarrow$ (3): The proof is similar to the proof of (2) $\Rightarrow$ (1) in Theorem G.5.3. For the reader's convenience we indicate the main steps.

Let

$$0 < \delta < \left\{ \sin(\omega), \frac{1}{eM} \right\},$$

where M is the R -bound of $\{tAS(t) : t \geq 0\}$. Then $\bigcup_{t>0} B(t, \delta t) = \Sigma_\eta$ with $\delta = \sin(\eta)$; note that $0 < \eta < \omega$. For any $z \in \Sigma_\eta$, choose $t = t(z) > 0$ such that $z \in B(t, \delta t)$. Then we can express $S(z)$ in terms of the Taylor expansion of $z \mapsto S(z)$ in a point $t > 0$:

$$S(z) = S(t) + \sum_{n \geq 1} \frac{1}{n!} (-A)^n S(t) (z - t)^n$$

$$= S(t) + \sum_{n \geq 1} \frac{n^n}{n!} \left(-(t/n)AS(t/n) \left(\frac{z}{t} - 1 \right) \right)^n.$$

Using Proposition 8.1.24 (with $I = \Sigma_\eta$) together with the inequality $n^n \leq n!e^n$, and noting that $|z - t| < \delta t$, we obtain

$$\mathcal{R}(\{S(z) : |\arg(z)| \leq \delta\}) \leq \mathcal{R}(\{S(t) : t \geq 0\}) + \sum_{n \geq 1} e^n M^n \delta^n.$$

The series on the right-hand side is convergent by the choice of δ .

(3) $\Rightarrow$ (1): Let $0 < \delta' < \delta$. The R -boundedness of the operators $\lambda R(\lambda, A)$ on the punctured lines $e^{\pm \frac{1}{2}\pi i \delta'} (\mathbb{R} \setminus \{0\})$ follows from Theorem 8.5.2 and the representation of the resolvent as the Laplace transform of the C_0 -semigroups $(S(e^{\pm i \delta'} t))_{t \geq 0}$. As in the proof of (1) $\Rightarrow$ (4), the R -boundedness in $\mathbb{C}\overline{\Sigma_{\frac{1}{2}\pi - \delta}}$ follows from this by Proposition 8.5.8. $\square$

10.3.b The main R -boundedness result

The main result of this subsection is formulated under two different geometric assumptions on X in terms of double random sums:

- (a) the *triangular contraction property* with associated constant Δ_X (see Definition 7.5.7);
- (b) the *Pisier contraction property* with associated constants α_X and $\alpha_X^\pm$ (see Definition 7.5.1 and Proposition 7.5.4).

Pisier's contraction property trivially implies the triangular contraction property. Every UMD space has the triangular contraction property (Theorem 7.5.9), and a Banach lattice has Pisier's contraction property if and only if it has finite cotype (Theorem 7.5.20).

Somewhat informally, the main result below states that

- (a) if X has the triangular contraction property and A has a bounded H^∞ -calculus, then A is R -sectorial;
- (b) if X has Pisier's contraction property and A has a bounded H^∞ -calculus, then the full H^∞ -calculus of A is R -bounded.

We recall the notation $M_{\sigma, A}^\infty$ for the boundedness constant of the $H^\infty(\Sigma_\sigma)$ -calculus of A .

Theorem 10.3.4 (R -sectoriality, R -boundedness of the H^∞ -calculus).

Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. If A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus, then:

- (1) for all $\sigma < \vartheta < \pi$, $0 < \delta < \vartheta - \sigma$, and $f \in H^1(\Sigma_\vartheta)$, the set

$$\{f(zA) : |\arg(z)| \leq \delta\}$$

is R -bounded, with R -bound at most $C_{\delta, \sigma, \vartheta} M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}$;

- (2) if X has the triangular contraction property, and A is densely defined, then A is R -sectorial with $\omega_R(A) \leq \omega_{H^\infty}(A)$, and for all $\omega_{H^\infty}(A) < \sigma < \vartheta < \pi$ the set

$$\{f(A) : f \in H^1(\Sigma_\vartheta), \|f\|_{H^1(\Sigma_\sigma)} \leq 1\}$$

is R -bounded, with R -bound at most $C_{\sigma,\vartheta}(M_{\sigma,A}^\infty)^2 \Delta_X$;

- (3) if X has Pisier's contraction property, and A is densely defined and has a dense range and a bounded $H^\infty(\Sigma_\eta)$ -calculus for some $\omega(A) < \eta < \sigma$, then the set

$$\{f(A) : f \in H^\infty(\Sigma_\sigma), \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$$

is R -bounded, with R -bound at most $C_{\eta,\sigma}(M_{\eta,A}^\infty)^2 \alpha_X^- \alpha_X^+$.

The attentive reader will notice that the assumption that A is densely defined and has dense range in (3) is redundant provided we interpret $f(A)$ as a bounded operator on $\overline{D(A) \cap R(A)}$. In the formulation as stated the operators $f(A)$ are defined on X .

The proof of this theorem requires some preparation. In Section 10.3.c we associate certain unconditional decompositions to the H^∞ -calculus of A . The proof of Theorem 10.3.4 will be given in Section 10.3.d.

Examples of R -bounded sets associated with a sectorial operator A

We continue with several examples of applications of Proposition 10.3.2 and Theorem 10.3.4.

Example 10.3.5. Let A be R -sectorial and let $\omega_R(A) < \sigma < \pi$. By Proposition 10.3.2, for all $\beta > \alpha > 0$ and $0 < \delta < \pi - \sigma$, the set

$$\{(zA)^\alpha (I + zA)^{-\beta} : |\arg(z)| \leq \delta\}$$

is R -bounded. These operators are to be interpreted as $f_{\alpha,\beta}(A)$ with

$$f_{\alpha,\beta}(\lambda) = \frac{\lambda^\alpha}{(1 + \lambda)^\beta}.$$

Similar interpretations apply below. Clearly, for $|\arg(z)| \leq \delta$ the functions $f_{\alpha,\beta}(z \cdot)$ belong to $H^1(\Sigma_\sigma)$ with norms $\|f_{\alpha,\beta}(z \cdot)\|_{H^1(\Sigma_\sigma)} \leq C_{\sigma+\delta}$, and therefore Proposition 10.3.2 gives the required result.

By the same reasoning, if $\omega_R(A) < \sigma < \frac{1}{2}\pi$, then for all $\alpha > 0$ and $0 < \delta < \frac{1}{2}\pi - \sigma$ the set

$$\{(zA)^\alpha e^{-zA} : |\arg(z)| \leq \delta\}$$

is R -bounded.

The above two sets are also R -bounded under the alternative assumption that A is a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus for some $\sigma \in (0, \pi)$; this time the result follows from Theorem 10.3.4(1).

The next example shows that the triangular contraction property cannot be omitted from Theorem 10.3.4(2).

Example 10.3.6. Let X be a Banach space. On the Rademacher space $\varepsilon(X) = \varepsilon^2(X)$ (see Section 6.3) we define the operator A by

$$A((x_n)_{n \geq 1}) := (2^n x_n)_{n \geq 1} \quad (10.20)$$

with its natural maximal domain consisting of those sequences $(x_n)_{n \geq 1} \in \varepsilon(X)$ for which $(2^n x_n)_{n \geq 1}$ belongs to $\varepsilon(X)$ again. By Kahane's contraction principle, A is sectorial of zero angle. Moreover, A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for any $0 < \sigma < \pi$, given by

$$f((x_n)_{n \geq 1}) := (f(2^n x_n)_{n \geq 1};$$

this easily follows by verifying the conditions of Theorem 10.2.14. If X has the triangular contraction property, then so does $\varepsilon(X)$, and therefore A is R -sectorial by Theorem 10.3.4(2).

Let now X be an arbitrary Banach space and suppose that the operator A defined by (10.20) is R -sectorial. We will show that X has the triangular contraction property. Let $M \geq 1$ be an arbitrary integer, set $\lambda_j := 2^{M(j+\frac{1}{2})}$ for $j \in \mathbb{Z}$, and let $(x_{n,k})_{k,n=1}^K$ be a sequence in X . Define the elements $x^{(k)} = (x_n^{(k)})_{n \geq 1} \in \varepsilon(X)$ by $x_{2^{Mn}}^{(k)} := x_{n,k}$ for $n \in \{1, \dots, K\}$ and $x_j^{(k)} := 0$ for the remaining coordinates. Then, by the assumed R -sectoriality of A ,

$$\begin{aligned} & \mathbb{E}' \mathbb{E} \left\| \sum_{k=1}^K \sum_{n=1}^K \varepsilon'_k \varepsilon_n \lambda_{n-k} (1 + \lambda_{n-k})^{-1} x_{n,k} \right\|_X^2 \\ &= \mathbb{E}' \left\| \sum_{k=1}^K \varepsilon'_k (1 + \lambda_k^{-1} A)^{-1} x^{(k)} \right\|_{\varepsilon(X)}^2 \\ &\leq (M_{\sigma,A}^R)^2 \mathbb{E}' \left\| \sum_{k=1}^K \varepsilon'_k x^{(k)} \right\|_{\varepsilon(X)}^2 \\ &= (M_{\sigma,A}^R)^2 \mathbb{E}' \mathbb{E} \left\| \sum_{k=1}^K \sum_{n=1}^K \varepsilon'_k \varepsilon_n x_{n,k} \right\|_X^2. \end{aligned}$$

Letting $M \rightarrow \infty$, we obtain

$$\mathbb{E}' \mathbb{E} \left\| \sum_{k=1}^K \sum_{n=1}^K \varepsilon'_k \varepsilon_n x_{n,k} \right\|_X^2 \leq (M_{\sigma,A}^R)^2 \mathbb{E}' \mathbb{E} \left\| \sum_{k=1}^K \sum_{n=1}^K \varepsilon'_k \varepsilon_n x_{n,k} \right\|_X^2.$$

This shows that X has the triangular projection property, with constant $\Delta_X \leq M_{\sigma,A}^R$.

Also Pisier's contraction property cannot be omitted in Theorem 10.3.4(3):

Example 10.3.7. As we have seen in Example 10.2.26, $A = -\Delta$ is sectorial and has a bounded H^∞ -calculus of zero angle on $L^p(\mathbb{R}^d; X)$ whenever X is a UMD space and $1 < p < \infty$. Theorem 10.3.4(3) implies that if in addition X has Pisier's contraction property, then for any $0 < \sigma < \pi$ the set

$$\{f(-\Delta) : f \in H^\infty(\Sigma_\sigma), \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\} \quad (10.21)$$

is R -bounded. (Alternatively, one can derive this from Corollary 8.3.23, where even more general Fourier multiplier operators are considered.)

Let now X be an arbitrary UMD Banach space and let again $1 < p < \infty$. For $s \in \mathbb{R}$ consider the functions $f_s(z) := z^{is}$. These functions are uniformly bounded in $H^\infty(\Sigma_\sigma)$ for each $0 < \sigma < \pi$, and therefore the operators $f_s(-\Delta) =: (-\Delta)^{is}$ are well defined and uniformly bounded on $L^p(\mathbb{R}^d; X)$. Up to a normalising constant, the set

$$\{(-\Delta)^{is} : s \in (0, 1]\} \quad (10.22)$$

is a subset of the set in (10.21). In Section 10.5 we will show that if the set in (10.22) is R -bounded, then X has Pisier's contraction property.

10.3.c Unconditional decompositions associated with A

In the setting of diagonal operators on Schauder bases, Proposition 10.2.28 revealed a clear connection between boundedness of H^∞ -calculus and unconditionality. We will now develop this idea more systematically by showing that the H^∞ -calculus of a general sectorial operator A can be understood in terms of unconditional “decompositions” associated the operator A . To explain this further, suppose that A has a bounded H^∞ -calculus on a Banach space X , let $\omega_{H^\infty}(A) < \sigma < \pi$, and consider a function $f \in H^1(\Sigma_\sigma)$ satisfying

$$\sum_{n \in \mathbb{Z}} f(2^n z) = 1, \quad z \in \Sigma_\sigma. \quad (10.23)$$

Such functions can be constructed by taking any $\psi \in H^1(\Sigma_\sigma)$ satisfying $\int_0^\infty \psi(t) \frac{dt}{t} = 1$, and putting

$$f(z) := \int_1^2 \psi(tz) \frac{dt}{t}. \quad (10.24)$$

Applying the Calderón reproducing formula of Proposition 10.2.15, in combination with the unconditionality result proved in Lemma 10.3.8 below the identity (10.23) will give an unconditionally convergent decomposition

$$x = \sum_{n \in \mathbb{Z}} f(2^n A)x, \quad x \in \overline{\mathcal{D}(A)} \cap \mathcal{R}(A).$$

The operators $f(2^n A)$ can thus be viewed as analogues to “coordinate projections”, the main difference being that the operators $f(2^k A)$ and $f(2^\ell A)$ are

no longer disjoint projections for $k \neq \ell$, but have some “overlap”. Reasoning along the same lines, for $g(z) := z(1+z)^{-1}$ the operators $g(2^n A) = 2^n A(I + 2^n A)^{-1} = A(2^{-n} + A)^{-1}$ can be viewed as analogues to “partial sum projections” (recall from Proposition 10.1.7 that $A(t+A)^{-1} \rightarrow I$ strongly on $\mathcal{R}(A)$). It turns out that this can be used to show that the family $\{g(2^n A) : n \in \mathbb{Z}\}$ is R -bounded if X has the triangular contraction property.

It will turn out that such results indeed hold for general operators A with a bounded H^∞ -calculus and will essentially follow from the unconditional convergence of the sums $\sum_{n \in \mathbb{Z}} f(2^n A)x$. The main technical tool for this is the following lemma.

Lemma 10.3.8 (Unconditionality). *Let A be a sectorial operator on a Banach space X with a bounded H^∞ -calculus. Let $\omega_{H^\infty}(A) < \sigma < \vartheta < \pi$. If $f \in H^1(\Sigma_\vartheta)$, then:*

(1) *for all finite subsets $F \subseteq \mathbb{Z}$ and scalars $|\alpha_j| \leq 1$, $j \in F$, we have*

$$\sup_{t>0} \left\| \sum_{j \in F} \alpha_j f(2^j t A) \right\| \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}, \quad (10.25)$$

where $M_{\sigma,A}^\infty$ is the boundedness constant of the $H^\infty(\Sigma_\sigma)$ -calculus of A ;

(2) *if, in addition, f satisfies*

$$\sum_{n \in \mathbb{Z}} f(2^n z) = 1, \quad z \in \Sigma_\sigma,$$

then

$$x = \sum_{n \in \mathbb{Z}} f(2^n A)x, \quad x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)},$$

with unconditional convergence of the sum.

Proof. (1): Consider the function $g : \Sigma_\sigma \rightarrow \mathbb{C}$,

$$g(z) := \sum_{j \in F} \alpha_j f(2^j t z),$$

with $t > 0$, $F \subseteq \mathbb{Z}$ finite, and $|\alpha_j| \leq 1$ for $j \in F$. By Proposition H.2.3 (applied to the functions $t \mapsto f(tz)$), g belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ with

$$\|g\|_{H^\infty(\Sigma_\sigma)} \leq K_{\vartheta-\sigma} \|f\|_{H^1(\Sigma_\vartheta)}. \quad (10.26)$$

From the boundedness of the $H^\infty(\Sigma_\sigma)$ -calculus of A we then obtain

$$\left\| \sum_{j \in F} \alpha_j f(2^j t A) \right\| = \|g(A)\| \leq M_{\sigma,A}^\infty \|g\|_{H^\infty(\Sigma_\sigma)} \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}.$$

(2): In view of the bound proved in (1) it suffices to consider elements $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Now (10.26) implies

$$\sum_{n=-M}^N |f(2^n z)| \leq K_{\vartheta-\sigma} \|f\|_{H^1(\Sigma_\vartheta)}, \quad z \in \Sigma_\sigma. \quad (10.27)$$

Let us write $x = \zeta(A)y = A(I + A)^{-2}y$. For $\omega_{H^\infty}(A) < \nu < \sigma$, the functions $f(2^n \cdot)\zeta(\cdot)$ belong to $H^1(\Sigma_\nu)$, and by dominated convergence and the properties of the functional calculus we obtain

$$\begin{aligned} \sum_{n=-M}^N f(2^n A)x &= \sum_{n=-M}^N f(2^n A)\zeta(A)y \\ &= \sum_{n=-M}^N \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(2^n z)R(z, A)\zeta(z)y \, dz \\ &\xrightarrow{M, N \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} R(z, A)\zeta(z)y \, dz = \zeta(A)y = x, \end{aligned}$$

with absolute convergence of the sum on the left hand side because of (10.27).

This proves the unconditional (and even absolute) convergence for $x \in \overline{D(A) \cap R(A)}$. To prove the unconditional convergence for $x \in \overline{D(A) \cap R(A)}$, let $\varepsilon > 0$ be arbitrary and fixed, and choose $y \in D(A) \cap R(A)$ such that $\|x - y\| < \varepsilon$. Since $y = \sum_{n \in \mathbb{Z}} f(2^n A)y$ converges unconditionally we may choose a finite set $F_\varepsilon \subseteq \mathbb{Z}$ such that for every finite set $F \subseteq \mathbb{Z}$ containing F_ε we have $\|y - \sum_{n \in F} f(2^n A)y\| < \varepsilon$. Now let $F \subseteq \mathbb{Z}$ be a finite set containing F_ε . Then from (10.25) we obtain

$$\begin{aligned} \left\| x - \sum_{n \in F} f(2^n A)x \right\| &\leq \|x - y\| + \left\| y - \sum_{n \in F} f(2^n A)y \right\| + \left\| \sum_{n \in F} f(2^n A)(y - x) \right\| \\ &\leq 2\varepsilon + K_{\vartheta-\sigma} M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)} \varepsilon, \end{aligned}$$

and the unconditional convergence follows. $\square$

Remark 10.3.9. For functions $f \in H^1(\Sigma_\vartheta)$ of the form (10.24) with $z \mapsto \psi(z) \log(z) \in H^1(\Sigma_\sigma)$ and $\omega(A) < \sigma < \vartheta$, part (2) of the lemma remains true for general sectorial operators A ; in this case, Proposition 10.2.5 and (10.12) give the following version of the bound in (1):

$$\left\| \sum_{k=m}^n f(2^k A)x \right\| \leq C_{\psi, \sigma} \|x\|, \quad x \in \overline{D(A) \cap R(A)},$$

uniformly with respect to $m, n \geq 1$, where $C_{\psi, \sigma}$ is the constant in the proof of Proposition 10.2.5.

10.3.d Proof of the main result

Proof of Theorem 10.3.4(1). Fix $f \in H^1(\Sigma_\vartheta)$ and $0 < \delta < \vartheta - \sigma$. We need to prove the following estimate

$$\mathcal{R}(\{f(zA) : |\arg(z)| \leq \delta\}) \leq C_{\delta, \sigma, \vartheta} M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}.$$

Let $\delta < \omega < \vartheta - \sigma$ and put $f_{\pm\omega}(z) := f(e^{\pm i\omega} z)$ for $z \in \Sigma_\sigma$. Since $f_{\pm\omega} \in H^1(\Sigma_{\vartheta-\omega})$ of norm $\|f_{\pm\omega}\|_{H^1(\Sigma_{\vartheta-\omega})} \leq \|f\|_{H^1(\Sigma_\vartheta)}$, Lemma 10.3.8 gives

$$\left\| \sum_{n \in F} \epsilon_n f_{\pm\omega}(2^n A) \right\| \leq K_{\vartheta-\sigma} M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}$$

for all finite $F \subseteq \mathbb{Z}$ and all finite sequences of modulus one scalars $(\epsilon_n)_{n \in F}$. In particular, the condition of Proposition 8.4.6(1) is satisfied, and therefore for all $t > 0$ we have

$$\mathcal{R}(\{f_{\pm\omega}(2^n t A) : n \in \mathbb{Z}\}) \leq K_{\vartheta-\sigma} M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}.$$

Now Proposition 8.5.8 gives

$$\mathcal{R}(\{f(zA) : |\arg(z)| \leq \delta\}) \leq 4K_{\vartheta-\sigma} \left(1 + \frac{1}{\pi} \tan\left(\frac{\delta}{\omega} \frac{1}{2}\pi\right)\right) M_{\sigma, A}^\infty \|f\|_{H^1(\Sigma_\vartheta)}.$$

Letting $\omega \uparrow \vartheta - \sigma$ the desired result is obtained, with constant $C_{\delta, \sigma, \vartheta} = 4K_{\vartheta-\sigma}(1 + \frac{1}{\pi} \tan(\frac{\delta}{\vartheta-\sigma} \frac{\pi}{2}))$. $\square$

For the proof of Theorem 10.3.4(2) we need some preparation.

Lemma 10.3.10. *The function $f : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$,*

$$f(z) = \frac{z^{1/2}}{(1+z)^{1/2}(1+2z)^{1/2}},$$

belongs to $H^1(\Sigma_\sigma)$, $0 < \sigma < \pi$, and

$$\|f\|_{H^1(\Sigma_\sigma)} \leq \frac{4}{\sin(\sigma \vee \frac{\pi}{2})}.$$

Proof. Let $r > 0$ and $0 < |\nu| \leq \sigma < \pi$ be arbitrary. By Lemma 10.2.4 we have $|1 + re^{i\nu}| \geq (r \vee 1) \sin(\sigma \vee \frac{1}{2}\pi)$ and therefore

$$|f(re^{i\nu})| \leq \frac{r^{1/2}}{(r \vee 1)^{1/2}((2r) \vee 1)^{1/2} \sin(\sigma \vee \frac{1}{2}\pi)} \leq \frac{r^{1/2}}{(r \vee 1) \sin(\sigma \vee \frac{1}{2}\pi)}.$$

This estimate easily implies

$$\int_0^\infty |f(te^{i\nu})| \frac{dt}{t} \leq \frac{4}{\sin(\sigma \vee \frac{1}{2}\pi)},$$

which is the required bound on the H^1 -norm of f . $\square$

Proof of Theorem 10.3.4(2). We begin by proving the R -boundedness of the set $\{(I + zA)^{-1} : |\arg(z)| \leq \delta\}$ for any fixed $0 < \delta < \pi - \sigma$. Choose ω and ϑ such that

$$\delta < \omega < \pi - \vartheta < \pi - \sigma.$$

The idea of the proof is to deduce this from Proposition 8.4.6(2) for a suitable family $(U_j)_{j \in \mathbb{Z}}$, using Proposition 8.5.8 to discretise the problem.

Fix $t > 0$ and $m, n \in \mathbb{Z}$ with $m < n$. Using the resolvent identity

$$(I + \lambda A)^{-1} - (I + \mu A)^{-1} = A(\mu - \lambda)(I + \lambda A)^{-1}(I + \mu A)^{-1}$$

and a telescoping argument, we obtain

$$\begin{aligned} & (I + 2^m t e^{\pm i\omega} A)^{-1} - (I + 2^n t e^{\pm i\omega} A)^{-1} \\ &= \sum_{j=m}^{n-1} (I + 2^j t e^{\pm i\omega} A)^{-1} - (I + 2^{j+1} t e^{\pm i\omega} A)^{-1} \\ &= \sum_{j=m}^{n-1} 2^j t e^{\pm i\omega} A (I + 2^j t e^{\pm i\omega} A)^{-1} (I + 2^{j+1} t e^{\pm i\omega} A)^{-1} \\ &= \sum_{j=m}^{n-1} f^2(2^j t e^{\pm i\omega} A) = \sum_{j=m}^{n-1} U_j^2, \end{aligned}$$

where f is as in Lemma 10.3.10 and $U_j := f(2^j t e^{\pm i\omega} A)$. Since for the function $f_{\pm\omega} = f(e^{\pm i\omega} \cdot)$, we have

$$\|f_{\pm\omega}\|_{H^1(\Sigma_\vartheta)} \leq \|f\|_{H^1(\Sigma_{\vartheta+\omega})} \leq C_{\vartheta+\omega},$$

where $C_{\vartheta+\omega} := 4/\sin((\vartheta + \omega) \vee \frac{\pi}{2})$, by Lemma 10.3.10. Therefore, by Lemma 10.3.8 (applied the rotated function $z \mapsto f(e^{\pm i\omega} z)$) and give

$$\sup_{t>0} \sup_{m<n} \sup_{|\epsilon_m|, \dots, |\epsilon_{n-1}|=1} \left\| \sum_{j=m}^{n-1} \epsilon_j U_j \right\| \leq C_{\vartheta+\omega} K_{\vartheta-\sigma} M_{\sigma,A}^\infty.$$

Hence by Proposition 8.4.6(2) with $U_n = V_n$ and $T_n = I$,

$$\mathcal{R}\left\{ \sum_{j=m}^{n-1} U_j^2 : m, n \in \mathbb{Z} \right\} \leq C_{\vartheta+\omega}^2 K_{\vartheta-\sigma}^2 (M_{\sigma,A}^\infty)^2 \Delta_X.$$

By the above identities, this gives the R -boundedness of the set

$$\mathcal{S}(t) := \left\{ (I + 2^m t e^{\pm i\omega} A)^{-1} - (I + 2^n t e^{\pm i\omega} A)^{-1} : m, n \in \mathbb{Z}, m < n \right\},$$

with the same R -bound. Next, since A is densely defined, $\lim_{m \rightarrow -\infty} (I + t 2^m e^{\pm i\omega} A)^{-1} x = x$ for all $x \in \mathcal{D}(A) = X$ by Proposition 10.1.7(1). Therefore the family

$$\{(I + 2^n t e^{\pm i\omega} A)^{-1} - I : n \in \mathbb{Z}\}$$

belongs to the strong operator closure of $\mathcal{S}(t)$ and is therefore R -bounded by Proposition 8.1.22, with the same R -bound. It follows that

$$\mathcal{R}(\{(I + 2^n t e^{\pm i\omega} A)^{-1} : n \in \mathbb{Z}\}) \leq 1 + K_{\vartheta-\sigma}^2 (M_{\sigma,A}^\infty)^2 \Delta_X.$$

So far, $t > 0$ was arbitrary and fixed. Now by Proposition 8.5.8,

$$M_{\pi-\delta,A}^R = \mathcal{R}(\{(I + zA)^{-1} : |\arg(z)| \leq \delta\}) \leq C_{\delta,\sigma} (M_{\sigma,A}^\infty)^2 \Delta_X,$$

where $C_{\delta,\sigma} = 4(1 + \frac{1}{\pi} \tan(\frac{\delta}{2}\pi)) (1 + K_{\vartheta-\sigma}^2 \Delta_X)$ (we may suppress the dependence on ϑ and ω , since they can be made depend on δ and σ). $\square$

For the proof of Theorem 10.3.4(3) we also need some preparation.

Lemma 10.3.11. *Let A be a sectorial operator on a Banach space X . Let $\omega(A) < \sigma < \pi$ and let $f \in H^1(\Sigma_\sigma)$ be given. For $\mu \in \mathbb{C}_{\overline{\Sigma_\sigma}}$ consider $r_\mu(z) := (\mu - z)^{-1}$. Then $r_\mu f \in H^1(\Sigma_\sigma)$ and*

$$(r_\mu f)(A) = R(\mu, A)f(A).$$

Proof. As in Proposition 10.2.6 let

$$\zeta_n(z) = \frac{n}{n+z} - \frac{1}{1+nz} = \frac{(n^2-1)z}{(n+z)(1+nz)}. \quad (10.28)$$

By the multiplicativity of the Dunford calculus (see Theorem 10.2.2)

$$\begin{aligned} (r_\mu f)(A) \zeta_n(A) &= (r_\mu f \zeta_n)(A) = (\zeta_n r_\mu)(A) f(A) \\ &= \zeta_n(A) R(\mu, A) f(A) = R(\mu, A) f(A) \zeta_n(A), \end{aligned}$$

where in the penultimate step we used Proposition 10.2.3 and in the last step the commutativity of the Dunford calculus. Now the required identity follows from the convergence property of the Dunford calculus. $\square$

Lemma 10.3.12. *Let $0 < \mu < \nu < \pi$. For all $0 < \alpha < 1$ and non-zero $z \in \partial\Sigma_\nu$ the function $\psi_{z,\alpha}(\lambda) := \lambda^\alpha / (z - \lambda)^{2\alpha}$ belongs to $H^1(\Sigma_\mu)$ and*

$$\|\psi_{z,\alpha}\|_{H^1(\Sigma_{\nu-\mu})} \leq C_{\nu-\mu,\alpha} |z|^{-\alpha}$$

with $C_{\nu-\mu,\alpha} = \frac{2}{\alpha} (\sin(\frac{1}{2}\pi \vee (\pi - \theta)))^{2\alpha}$.

Proof. For all $|\theta| < \mu$ we have

$$\begin{aligned} \int_0^\infty |\psi_{z,\alpha}(re^{i\theta})| \frac{dr}{r} &= \int_0^\infty \frac{r^\alpha}{|z - re^{i\theta}|^{2\alpha}} \frac{dr}{r} \\ &\leq |z|^{-\alpha} \int_0^\infty \frac{\rho^\alpha}{|1 - \rho e^{-i(\nu-\mu)}|^{2\alpha}} \frac{d\rho}{\rho} =: c_{\nu-\mu,\alpha} |z|^{-\alpha}, \end{aligned}$$

where, by using Lemma 10.2.4,

$$c_{\theta,\alpha} \leq \frac{1}{(\sin(\frac{1}{2}\pi \vee (\pi - \theta)))^{2\alpha}} \int_0^\infty \frac{\rho^\alpha}{(1 \vee \rho^{2\alpha})} \frac{d\rho}{\rho}$$

and

$$\int_0^\infty \frac{\rho^\alpha}{(1 \vee \rho^{2\alpha})} \frac{d\rho}{\rho} = \int_0^1 \rho^\alpha \frac{d\rho}{\rho} + \int_1^\infty \rho^{-\alpha} \frac{d\rho}{\rho} = \frac{2}{\alpha}.$$

□

Lemma 10.3.13. *Let A be a sectorial operator on a Banach space X . Let $\omega(A) < \sigma < \pi$ and let $f \in H^1(\Sigma_\sigma)$ be given. Let $\phi_z(\lambda) := \lambda^{1/2}/(z - \lambda)$. Then for all $\omega(A) < \nu < \sigma$ we have*

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} z^{1/2} f(z) \phi_z(A) \frac{dz}{z}. \quad (10.29)$$

Before turning to the proof we must justify the well-definedness of the operators $\phi_z(A)$ and the convergence of integral on the right-hand side. By Lemma 10.3.12, the operators $\phi_z(A)$ are well defined through the Dunford calculus and the family $\{z^{1/2}\phi_z(A) : z \in \partial\Sigma_\nu\}$ is uniformly bounded. This, in turn, combined with the fact that $f \in H^1(\Sigma_\sigma)$, shows that the integral in the statement of the lemma is absolutely convergent.

Comparing (10.29) with the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z) [zR(z, A)] \frac{dz}{z}$$

we see what the lemma achieves: it allows us to replace the bounded function $z \mapsto zR(z, A)$ in the Dunford integral by another function, namely the function $z \mapsto z^{1/2}\phi_z(A)$. The improvement consists of the fact that the operators $z^{1/2}\phi_z(A)$ are obtained through the Dunford calculus of A , whereas the operators $zR(z, A)$ are not (as $\lambda \mapsto z/(\lambda - z)$ does not belong to $H^1(\Sigma_\omega)$ for any $0 < \omega < \pi$).

Proof of Lemma 10.3.13. Let $\omega(A) < \nu < \sigma < \pi$ be as in the statement of the lemma and fix $\omega(A) < \mu < \nu$. Let ζ_n , $n \geq 1$, be the functions in (10.28) and put

$$\eta_n^\pm(\lambda) := \lambda^{\pm 1/2} \zeta_n(\lambda), \quad \lambda \in \Sigma_\mu.$$

By the multiplicativity of the Dunford calculus,

$$f(A) \zeta_n(A)^2 = (f \zeta_n^2)(A) (\eta^+ \cdot (\eta^- f))(A) = \eta_n^+(A) (\eta_n^- f)(A).$$

For $z \in \partial\Sigma_\nu$ we may apply Lemma 10.3.11 to see that

$$\eta_n^+(A) R(z, A) = (\eta_n^+ r_z)(A).$$

Furthermore, it follows from the identity

$$\eta_n^+(\lambda)r_z(\lambda) = \lambda^{1/2}\zeta_n(\lambda)(z - \lambda)^{-1} = \phi_z(\lambda)\zeta_n(\lambda)$$

and the multiplicativity of the Dunford calculus that

$$(\eta_n^+r_z)(A) = \phi_z(A)\zeta_n(A), \quad z \in \partial\Sigma_\nu.$$

It follows that

$$\begin{aligned} f(A)\zeta_n(A)^2 &= \frac{1}{2\pi i}\eta_n^+(A) \int_{\partial\Sigma_\nu} \eta_n^-(z)f(z)R(z, A) \, dz \\ &= \frac{1}{2\pi i}\eta_n^+(A) \int_{\partial\Sigma_\nu} z^{-1/2}\zeta_n(z)f(z)R(z, A) \, dz \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} z^{-1/2}\zeta_n(z)f(z)(\eta_n^+r_z)(A) \, dz \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \zeta_n(z)[z^{1/2}f(z)\phi_z(A)]\zeta_n(A) \frac{dz}{z}. \end{aligned}$$

Letting $n \rightarrow \infty$, (10.29) follows from the convergence property of the Dunford calculus and the dominated convergence theorem. $\square$

Proof of Theorem 10.3.4(3). We must prove the R -boundedness of the set

$$\mathcal{S} := \{f(A) : f \in H^\infty(\Sigma_\sigma), \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\},$$

along with the stated bound for its R -boundedness constant.

Fix $\omega_{H^\infty}(A) < \eta < \mu < \nu < \sigma$ and first consider a function $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. Let $\phi_z(\lambda) = \lambda^{1/2}/(z - \lambda)$ as in Lemma 10.3.13. By that lemma,

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} z^{1/2}f(z)\phi_z(A) \frac{dz}{z} \\ &= \sum_{\epsilon=\pm 1} \frac{1}{2\pi i} \epsilon e^{-\epsilon i\nu/2} \int_0^\infty f(e^{-\epsilon i\nu}t)\phi_{e^{-\epsilon i\nu}(t^{-1}A)} \frac{dt}{t} \\ &= \sum_{j \in \mathbb{Z}} \sum_{\epsilon=\pm 1} \frac{1}{2\pi i} \epsilon e^{-\epsilon i\nu/2} \int_1^2 f(e^{-\epsilon i\nu}2^j t)\phi_{e^{-\epsilon i\nu}(t^{-1}2^{-j}A)} \frac{dt}{t}. \end{aligned}$$

Fix $f_1, \dots, f_N \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ with $\|f_n\|_\infty \leq 1$ and $x_1, \dots, x_N \in X$. Introducing

$$U_j(t, \epsilon) := \phi_{e^{-\epsilon i\nu}(t^{-1}2^{-j}A)}^{1/2}$$

we can write

$$\sum_{n=1}^N \varepsilon_n f_n(A)x_n = \sum_{j \in \mathbb{Z}} \sum_{\epsilon=\pm 1} \frac{1}{2\pi i} \epsilon e^{-\epsilon i\nu/2} \int_1^2 \sum_{n=1}^N \varepsilon_n f_n(e^{-\epsilon i\nu}2^j t) U_j^2(t, \epsilon) x_n \frac{dt}{t},$$

where $(\varepsilon_n)_{n \in \mathbb{Z}}$ is a Rademacher sequence. Therefore,

$$\begin{aligned} & \left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n f_n(A) x_n \right\|^2 \right)^{1/2} \\ & \leq \frac{\log(2)}{\pi} \sup_{t>0} \sup_{k \geq 1} \max_{\epsilon = \pm 1} \left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n \sum_{|j| \leq k} f_n(e^{-\epsilon i \nu} 2^j t) U_j^2(t, \epsilon) x_n \right\|^2 \right)^{1/2} \\ & \leq \frac{\log(2)}{\pi} \sup_{t>0} \max_{\epsilon = \pm 1} \mathcal{R}(\mathcal{S}(t, \epsilon)) \left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n x_n \right\|^2 \right)^{1/2}, \end{aligned}$$

where for $t > 0$ and $\epsilon \in \{-1, 1\}$ the operator family $\mathcal{S}(t, \epsilon)$ is given by

$$\mathcal{S}(t, \epsilon) := \left\{ \sum_{|j| \leq k} f_n(e^{-\epsilon i \nu} 2^j t) U_j^2(t, \epsilon) : n \geq 1, k \geq 1 \right\}.$$

It remains to estimate $\mathcal{R}(\mathcal{S}(t, \epsilon))$ with a bound that is uniform in $t > 0$ and $\epsilon \in \{-1, 1\}$. To do so we will apply Proposition 8.4.6(3), for fixed $t > 0$, to the operators $U_n = V_n := U_n(t, \epsilon)$ and the set

$$\mathcal{T} := \{f_n(z) : z \in \Sigma_\sigma, n \in \mathbb{N}\} \subseteq \{cI : |c| \leq 1\},$$

which is R -bounded with constant 1, by the Kahane contraction principle.

To check the conditions on U_n needed to apply Proposition 8.4.6(3), we apply Lemma 10.3.8, which gives the bound

$$\sup_{t>0} \sup_{m < n} \sup_{|\epsilon_m|, \dots, |\epsilon_{n-1}|=1} \left\| \sum_{j=m}^{n-1} \epsilon_j U_j(t, \epsilon) \right\| \leq K_{\mu-\eta} M_{\eta,A}^\infty \|\phi_{e^{-\epsilon i \nu}}^{1/2}\|_{H^1(\Sigma_\mu)},$$

with $\|\phi_{e^{-\epsilon i \nu}}^{1/2}\|_{H^1(\Sigma_\mu)} \leq C_{\nu-\mu, 1/4}$ by Lemma 10.3.12, with the constant as in the lemma. Thanks to this bound we are in a position to apply Proposition 8.4.6(3) and obtain

$$\mathcal{R}(\mathcal{S}(t, \epsilon)) \leq K_{\mu-\eta}^2 (M_{\eta,A}^\infty)^2 C_{\nu-\mu, \frac{1}{4}}^2 \alpha_X^- \alpha_X^+.$$

Setting $\mathcal{S}_1 := \{f(A) : f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma), \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$, this proves that $\mathcal{S}_1$ is R -bounded and

$$\mathcal{R}(\mathcal{S}_1) \leq \frac{\log(2)}{\pi} \sup_{t>0} \sup_{\epsilon \in \{-1, 1\}} \mathcal{R}(\mathcal{S}(t, \epsilon)) \leq K_{\mu-\eta}^2 (M_{\eta,A}^\infty)^2 C_{\nu-\mu, 1/4}^2 \alpha_X^- \alpha_X^+.$$

Finally, since $f(A)\zeta_m(A)x \rightarrow f(A)x$ for all $x \in X = \overline{\mathbf{D}(A)} \cap \mathbf{R}(A)$, by Proposition 8.1.22 $\mathcal{S}$ is contained in the closure of $\mathcal{S}_1$ in the strong operator topology, and hence $\mathcal{S}$ is R -bounded with the same bound. $\square$

Remark 10.3.14. If we do not assume that A has dense range and dense domain in Theorem 10.3.4(3), then the above proof still gives the R -boundedness of $\mathcal{S}_1$ with the bound as stated.

The following simple corollary is of independent interest.

Corollary 10.3.15. *Let A be a sectorial operator on a Banach space X . Consider the function $\phi_z(\lambda) = \lambda^{1/2}/(z - \lambda)$. If, for some $\omega(A) < \nu < \pi$ we have*

$$\int_{\partial\Sigma_\nu} |z^{1/2}\langle\phi_z(A)x, x^*\rangle| \frac{|dz|}{|z|} < \infty$$

for all $x \in X$ and $x^ \in X^*$, then A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for all $\nu < \sigma < \pi$.*

Proof. It has already been observed in Lemma 10.3.13 the operators $\phi_z(A)$ are well defined by the Dunford calculus of A and the operators $z^{1/2}\phi_z(A)$ are uniformly bounded as z ranges over $\partial\Sigma_\nu$.

By a standard closed graph argument there exists a constant $C \geq 0$ such that

$$\int_{\partial\Sigma_\nu} |z^{1/2}\langle\phi_z(A)x, x^*\rangle| \frac{|dz|}{|z|} \leq C\|x\|\|x^*\|$$

for all $x \in X$ and $x^* \in X^*$.

Let $\omega(A) < \nu < \sigma < \pi$, and let $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ be given. By Lemma 10.3.13, for $x \in \overline{D(A)} \cap \overline{R(A)}$ we have

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} z^{1/2} f(z) \phi_z(A) \frac{dz}{z}.$$

Hence for all $x \in X$ and $x^* \in X^*$,

$$\begin{aligned} |\langle f(A)x, x^* \rangle| &= \frac{1}{2\pi} \|f\|_{H^\infty(\Sigma_\sigma)} \int_{\partial\Sigma_\nu} |z^{1/2}\langle\phi_z(A)x, x^*\rangle| \frac{|dz|}{|z|} \\ &\leq C\|x\|\|x^*\|\|f\|_{H^\infty(\Sigma_\sigma)}. \end{aligned}$$

Hence $\|f(A)x\| \leq \frac{C}{2\pi} \|f\|_{H^\infty(\Sigma_\sigma)} \|x\|$. This shows that condition (2) of Proposition 10.2.11 is satisfied. It follows that A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus with constant $M_{A,\nu}^\infty \leq \frac{C}{2\pi} / \sin(\sigma \vee \frac{1}{2}\pi)$ (recall that this constant has been defined as the best constant in condition (2) of this proposition). $\square$

10.4 Square functions and H^∞ -calculus

From its early days, the topic of H^∞ -calculus has been intimately connected with square functions. As we will see, the boundedness of suitable square functions associated with a sectorial operator imply that the operator has

a bounded H^∞ -calculus and vice versa. Such results were originally proved for continuous square functions in a Hilbert space setting and subsequently generalised to Banach spaces with finite cotype. A typical example of a square function estimate is the equivalence of norms

$$\left\| \left(\int_0^\infty \left| t^k \frac{d^k}{dt^k} e^{-t\Delta} f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \approx \|f\|_{L^p(\mathbb{R}^d)}, \quad (10.30)$$

valid for all $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$ and integers $k \geq 1$. The expression on the left-hand side is the classical Littlewood-Paley g -function norm. As we will see, the equivalence of norms (10.30) will be a simple consequence of the abstract equivalence of norms for continuous square functions associated with the operator $A = -\Delta$.

10.4.a Discrete square functions

We start with a study of discrete square function estimates. Apart from their intrinsic interest they have the additional advantage over their continuous analogues that they can be stated and proved without any geometrical assumptions on the Banach space X . In contrast, the treatment of continuous square functions requires X to have finite cotype in most of the results.

Definition 10.4.1 (Discrete square functions). *Let A be a sectorial operator on a Banach space X , let $\omega(A) < \sigma < \pi$, and consider a function $\phi \in H^1(\Sigma_\sigma)$. Let $(\varepsilon_n)_{n \in \mathbb{Z}}$ be a Rademacher sequence. The discrete square function norm, associated with A and ϕ , of a vector $x \in X$ is defined by*

$$\|x\|_{\phi,A} := \sup_{t>0} \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(2^n t A) x \right\|^2 \right)^{1/2}$$

whenever this expression is finite.

The supremum over $t > 0$ can be replaced by a supremum by $t \in [1, 2]$, as can be seen by writing $t = s2^n$ with $s \in [1, 2]$ and $n \in \mathbb{Z}$. The parameter t may seem awkward at first sight and can indeed be avoided under suitable assumptions (see Proposition 10.4.8 below).

Remark 10.4.2. In the setting of Definition 10.4.1, for all $x \in D(A) \cap R(A)$ the square function $\|x\|_{\phi,A}$ is finite. To see this, let $\omega(A) < \nu < \sigma$. Writing $x = \zeta(A)y$ with $\zeta(z) = z(1+z)^{-2}$, so $\zeta(A) = A(I+A)^{-2}$ by Proposition 10.2.3, we estimate the Dunford integral:

$$\|\phi(2^n t A)x\| \leq \frac{M_{\nu,A}}{2\pi} \int_{\partial \Sigma_\nu} |\phi(2^n tz)| |\zeta(z)| \|y\| \left| \frac{dz}{z} \right|.$$

Summing over $n \in \mathbb{Z}$ and using $\lim_{\nu \rightarrow \sigma} M_{\nu,A} = M_{\sigma,A}$ gives

$$\sum_{n \in \mathbb{Z}} \|\phi(2^n t A)x\| \leq \frac{K_{\sigma-\nu}}{\pi} M_{\sigma,A} \|\phi\|_{H^1(\Sigma_\sigma)} \|\zeta\|_{H^1(\Sigma_\sigma)} \|y\|,$$

where $K_\vartheta = (\frac{4}{\pi \log(2)} \vee \frac{2}{\pi \vartheta})^{1/p}$ is the constant of Proposition H.2.3 which we applied to the function $z \mapsto \phi(e^{\pm i\nu} z)$ on the sector $\Sigma_{\sigma-\nu}$.

Remark 10.4.3. The square function $\|x\|_{\phi,A}$ of Definition 10.4.1 was introduced using the dyadic points 2^n , $n \in \mathbb{Z}$. All results in this subsection easily remain true if the number 2 is replaced by any real number $q \in (1, \infty)$; the constants in the corresponding estimates will then also depend on the value of q .

The first main result of this section is the following square function bound for operators with a bounded H^∞ -calculus. Recall that the boundedness constant of the $H^\infty(\Sigma_\sigma)$ -calculus of an operator A is denoted by $M_{\sigma,A}^\infty$; this constant has been defined as the least admissible constant C in order that the inequality

$$\|f(A)x\| \leq C \|hf\|_{H^\infty(\Sigma_\sigma)} \|x\|$$

be satisfied for all $f \in H^\infty(\Sigma_\sigma)$ and $x \in \overline{\mathcal{D}(A)} \cap \mathcal{R}(A)$.

Theorem 10.4.4 (H^∞ -calculus implies square function bounds). *Let A be a sectorial operator with a bounded H^∞ -calculus on a Banach space X and let $\omega_{H^\infty}(A) < \sigma < \vartheta < \pi$. For all $\phi \in H^1(\Sigma_\vartheta)$ the following assertions hold:*

(1) *for all $x \in X$ we have*

$$\|x\|_{\phi,A} \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|$$

(2) *if A is densely defined, then for all $x^* \in X^*$ we have*

$$\|x^*\|_{\phi,A^*} \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x^*\|;$$

(3) *if A is densely defined and $\phi \neq 0$, then for all $x \in \overline{\mathcal{R}(A)}$ we have*

$$\|x\| \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty C_\phi \|x\|_{\phi,A},$$

$$\text{where } C_\phi = \|\phi\|_{H^1(\Sigma_\vartheta)} / \|\phi\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^2.$$

Note that any function $\phi \in H^1(\Sigma_\vartheta)$ restricts to a function in $H^\infty(\Sigma_{\vartheta'})$ for all $0 < \vartheta' < \vartheta$ by Proposition H.2.4 and hence to a function in $H^2(\Sigma_{\vartheta'})$. In particular, $t \mapsto \phi(t)$ belongs $L^2(\mathbb{R}_+, \frac{dt}{t})$ (and is non-zero since ϕ is non-zero as a function in $H^1(\Sigma_\vartheta)$).

Proof. (1): By Lemma 10.3.8, for all $x \in X$ we have

$$\|x\|_{\phi,A} \leq \sup_{t>0} \sup_{N \geq 1} \sup_{|\epsilon_{-N}|, \dots, |\epsilon_N|=1} \left\| \sum_{|n| \leq N} \epsilon_n \phi(2^n t A)x \right\|$$

$$\leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|.$$

Taking $\epsilon = \epsilon(\omega)$ and averaging with respect to $\omega \in \Omega$ we obtain (1).

(2): This follows from the previous case applied to the adjoint operator A^* , which has an H^∞ -calculus by Proposition 10.2.20.

(3): Fix any $\omega_{H^\infty}(A) < \sigma < \vartheta < \pi$. Let $\psi \in H^1(\Sigma_\vartheta)$ be given by $\psi(z) := c_\phi^{-1} \overline{\phi(\bar{z})}$ with $c_\phi := \int_0^\infty |\phi(t)|^2 \frac{dt}{t}$. Analyticity of ψ follows by considering its Taylor series. For all $x \in D(A) \cap R(A)$ and $x^* \in X^*$, from the Calderón reproducing formula (Proposition 10.2.5) we obtain

$$\begin{aligned} |\langle x, x^* \rangle| &= \left| \int_0^\infty \langle \phi(tA) \psi(tA) x, x^* \rangle \frac{dt}{t} \right| \\ &= \left| \sum_{n \in \mathbb{Z}} \int_1^2 \langle \phi(2^n tA) \psi(2^n tA) x, x^* \rangle \frac{dt}{t} \right| \\ &\leq \liminf_{N \rightarrow \infty} \int_1^2 \left| \sum_{|n| \leq N} \langle \phi(2^n tA) \psi(2^n tA) x, x^* \rangle \right| \frac{dt}{t} \\ &\leq \sup_{N \geq 1} \sup_{t \in [1,2]} \left| \sum_{|n| \leq N} \langle \phi(2^n tA) \psi(2^n tA) x, x^* \rangle \right|. \end{aligned}$$

Randomising the sum in the last expression using a Rademacher sequence $(\varepsilon_n)_{n \in \mathbb{Z}}$ and noting that complex conjugate sequence $(\bar{\varepsilon}_n)_{n \in \mathbb{Z}}$ is a Rademacher sequence as well, we obtain

$$\begin{aligned} |\langle x, x^* \rangle| &\leq \sup_{N \geq 1} \sup_{t \in [1,2]} \left| \mathbb{E} \left\langle \sum_{|n| \leq N} \varepsilon_n \phi(2^n tA) x, \sum_{|n| \leq N} \bar{\varepsilon}_n \psi(2^n tA^*) x^* \right\rangle \right| \\ &\leq \|x\|_{\phi,A} \|x^*\|_{\psi,A} \\ &\leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty c_\phi^{-1} \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|_{\phi,A^*} \|x^*\| \end{aligned}$$

by applying (2) in the last step. Taking the supremum over all $x^* \in X^*$ of norm ≤ 1 , we infer that

$$\|x\| \leq K_{\vartheta-\sigma} M_{\sigma,A}^\infty c_\phi^{-1} \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|_{\phi,A}.$$

This proves the desired bound for elements $x \in D(A) \cap R(A)$. By an approximating argument using (1) it follows that the estimate can be extended to $\overline{D(A) \cap R(A)} = \overline{R(A)}$ (by Proposition 10.1.8 and the assumption that $D(A)$ is dense in X). $\square$

Remark 10.4.5. Let A be a sectorial operator and consider a non-zero $\phi \in H^1(\Sigma_\vartheta)$, where $\omega(A) < \vartheta < \pi$. Inspecting the proof of Theorem 10.4.4, we see that its conclusions hold if we have

$$\sup_{t>0} \sup_{N \geq 1} \sup_{|\epsilon_{-N}|, \dots, |\epsilon_N|=1} \left\| \sum_{|n| \leq N} \epsilon_n \phi(2^n tA) \right\| < \infty.$$

Indeed, the assumption that A has a bounded H^∞ -calculus was only used to deduce this inequality via Lemma 10.3.8.

Our next aim is to prove a converse to Theorem 10.4.4: Theorem 10.4.9 below states that square function bounds together with R -sectoriality imply the boundedness of the H^∞ -calculus with $\omega_{H^\infty}(A) \leq \omega_R(A)$.

We need a preliminary result which allows us to compare the square functions $\|\cdot\|_{\phi,A}$ associated with different choices of ϕ . For the second part of the statement we recall from the discussion preceding Proposition 10.2.11 (see (10.14)) that for $x \in D(A) \cap R(A)$ and $f \in H^\infty(\Sigma_\sigma)$ one may define $f(A)x \in \overline{D(A) \cap R(A)}$ by regularisation with the function $\zeta(z) := z(1+z)^{-2}$ through

$$f(A)x := (f\zeta)(A)y$$

whenever $y \in X$ satisfies $\zeta(A)y = x$.

Recall the notation $M_{\sigma,A}^R$ for the R -bound of the set $\{zR(z, A) : |\arg(z)| < \sigma\}$.

Proposition 10.4.6 (Equivalence of discrete square functions). *Let A be an R -sectorial operator on a Banach space X and let $\omega_R(A) < \nu < \sigma < \pi$. Fix arbitrary $\phi, \psi \in H^1(\Sigma_\sigma)$ with $\psi \neq 0$. Then the following assertions hold:*

(1) *for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $x \in X$ we have*

$$\|f(A)x\|_{\phi,A} \leq C_{\phi,\psi} (M_{\sigma,A}^R)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|x\|_{\psi,A};$$

(2) *the same estimate holds for all $f \in H^\infty(\Sigma_\sigma)$ and $x \in D(A) \cap R(A)$, provided one interprets $f(A)x$ as explained above;*

(3) *if both ϕ and ψ are non-zero, then for all $x \in D(A) \cap R(A)$ we have an equivalence of homogeneous norms $\|x\|_{\phi,A} \approx \|x\|_{\psi,A}$, and moreover*

$$\|x\|_{\phi,A} \leq C_{\phi,\psi} (M_{\sigma,A}^R)^2 \|x\|_{\psi,A}.$$

The proof of Proposition 10.4.6 is based on following lemma. Recall that an operator-valued function is said to be *strongly measurable* if all of its orbits are strongly measurable.

Lemma 10.4.7. *Let $K, L : \mathbb{R}_+ \rightarrow \mathcal{L}(X)$ be bounded and strongly measurable functions and let $h \in L^1(\mathbb{R}_+, \frac{dt}{t})$. Assume that*

$$K(t)x = \int_0^\infty h(ts)L(s)x \frac{ds}{s}, \quad x \in X, \quad t > 0. \quad (10.31)$$

Then for all $x \in X$ we have

$$\sup_{t>0} \sup_{N \geq 1} \mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n K(2^n t)x \right\|^2 \leq \|h\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}^2 \sup_{t>0} \sup_{N \geq 1} \mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n L(2^n t)x \right\|^2.$$

Proof. In order to avoid convergence issues, we first replace h by $h_r := \mathbf{1}_{[r, 1/r]}h$ and L by $L_r := \mathbf{1}_{[r, 1/r]}L$ with $r \in (0, 1)$. Defining K_r as (10.31) with L replaced

by L_r , we have $K_r(t)x \rightarrow K(t)x$ for all $t \in (0, \infty)$ and $x \in X$. Moreover, for all $n \in \mathbb{Z}$ and $t > 0$,

$$K_r(2^n t)x = \int_1^2 \sum_{j \in \mathbb{Z}} h_r(2^{n+j} ts) L_r(2^j s)x \frac{ds}{s}.$$

By Kahane's contraction principle (see Theorem 6.1.13),

$$\begin{aligned} & \left\| \sum_{|n| \leq N} \varepsilon_n K_r(2^n t)x \right\|_{L^2(\Omega; X)} \\ & \leq \left\| \sum_{n \in \mathbb{Z}} \varepsilon_n K_r(2^n t)x \right\|_{L^2(\Omega; X)} \\ & \leq \int_1^2 \left\| \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \varepsilon_n h_r(2^{n+j} ts) L_r(2^j s)x \right\|_{L^2(\Omega; X)} \frac{ds}{s} \\ & = \int_1^2 \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon_{k-j} h_r(2^k ts) L_r(2^j s)x \right\|_{L^2(\Omega; X)} \frac{ds}{s} \\ & \leq \sum_{k \in \mathbb{Z}} \int_1^2 \left\| \sum_{j \in \mathbb{Z}} \varepsilon_{k-j} h_r(2^k ts) L_r(2^j s)x \right\|_{L^2(\Omega; X)} \frac{ds}{s} \\ & \leq \sum_{k \in \mathbb{Z}} \int_1^2 |h_r(ts 2^k)| \frac{ds}{s} \left(\sup_{s \in [1, 2]} \left\| \sum_{j \in \mathbb{Z}} \varepsilon_{k-j} L_r(2^j s)x \right\|_{L^2(\Omega; X)} \right) \\ & \leq \int_0^\infty |h(s)| \frac{ds}{s} \sup_{s \in [1, 2]} \sup_{N \geq 1} \left\| \sum_{|j| \leq N} \varepsilon_j L_r(2^j s)x \right\|_{L^2(\Omega; X)}. \end{aligned}$$

Again by the Kahane contraction principle,

$$\left\| \sum_{|j| \leq N} \varepsilon_j L_r(2^j s)x \right\|_{L^2(\Omega; X)} \leq \left\| \sum_{|j| \leq N} \varepsilon_j L(2^j s)x \right\|_{L^2(\Omega; X)}.$$

The assertion now follows by combining the estimates and letting $r \downarrow 0$. $\square$

Proof of Proposition 10.4.6. We only need to prove assertions (1) and (2), as (3) follows from (2) by taking $f \equiv 1$.

We present the proof for general $x \in X$ and functions $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$; the reader may follow the lines of the proof to check that the same proof applied almost *verbatim* to the case where $x \in D(A) \cap R(A)$ and $f \in H^\infty(\Sigma_\sigma)$; the only difference is that the multiplicativity of the Dunford calculus in (10.32) must now be invoked with $f\zeta$ instead of f . This necessitates choosing $x \in D(A) \cap R(A)$.

Let $\xi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ be an arbitrary non-zero function. Let $\eta \in H^1(\Sigma_\sigma)$ be defined by

$$\eta(z) := c_{\phi, \psi}^{-1} \overline{\psi(\bar{z}) \xi(\bar{z})}, \quad \text{where } c_{\phi, \psi} := \int_0^\infty |\psi(t)|^2 |\xi(t)|^2 \frac{dt}{t},$$

so that $\int_0^\infty \xi(t)\eta(t)\psi(t) \frac{dt}{t} = 1$ (the integral defining $c_{\phi,\psi}$ is finite by an argument similar to the one in the proof of Theorem 10.4.4. Analyticity of η follows by considering its Taylor series. Then

$$\int_0^\infty \xi(tz)\eta(tz)\psi(tz) \frac{dt}{t} = 1, \quad z \in \Sigma_\sigma,$$

which is clear for $z \in \mathbb{R} \setminus \{0\}$ and then in general by analytic continuation.

Let $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. By Fubini's theorem and the multiplicativity of the Dunford calculus

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z)R(z, A) dz \\ &= \frac{1}{2\pi i} \int_0^\infty \int_{\partial\Sigma_\nu} f(z)\xi(tz)\eta(tz)\psi(tz)R(z, A) dz \frac{dt}{t} \\ &= \int_0^\infty f(A)\xi(tA)\eta(tA)\psi(tA) \frac{dt}{t}. \end{aligned} \quad (10.32)$$

For $z \in \Sigma_\sigma$ set

$$U(z) := \int_0^\infty \xi(tz)f(A)\eta(tA)\psi(tA) \frac{dt}{t}, \quad L(z) := zR(z, A)U(z).$$

An easy estimate using (10.9) shows that the operators $U(z)$ thus defined are uniformly bounded, and by Fubini's theorem

$$\begin{aligned} f(A)\phi(sA)x &= \int_0^\infty f(A)\xi(tA)\eta(tA)\psi(tA)\phi(sA)x \frac{dt}{t} \\ &= \int_0^\infty \left(\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \xi(tz)\phi(sA)R(z, A) dz \right) f(A)\eta(tA)\psi(tA)x \frac{dt}{t} \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \phi(sA)R(z, A) \left(\int_0^\infty \xi(tz)f(A)\eta(tA)\psi(tA)x \frac{dt}{t} \right) dz \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \phi(sA)R(z, A)U(z)x dz \\ &= \frac{1}{2\pi i} \sum_{\epsilon=\pm 1} \int_0^\infty \epsilon \phi(e^{-i\epsilon\nu s t})L(e^{-i\epsilon\nu t})x \frac{dt}{t} \\ &= \frac{1}{2\pi i} \sum_{\epsilon=\pm 1} \int_0^\infty \epsilon \phi(e^{-i\epsilon\nu t})L(e^{-i\epsilon\nu s^{-1}t})x \frac{dt}{t}. \end{aligned} \quad (10.33)$$

Hence by Lemma 10.4.7,

$$\begin{aligned} \|f(A)x\|_{\phi, A} &= \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n f(A)\phi(2^n t A)x \right\|_{L^2(\Omega; X)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\pi} \|\phi\|_{H^1(\Sigma_\sigma)} \sup_{\epsilon \in \{-1,1\}} \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n L(e^{-i\epsilon\nu} 2^n t) x \right\|_{L^2(\Omega; X)} \\
&\leq \frac{1}{\pi} \|\phi\|_{H^1(\Sigma_\sigma)} M_{\nu, A}^R \sup_{\epsilon \in \{-1,1\}} \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n U(e^{-i\epsilon\nu} 2^n t) x \right\|_{L^2(\Omega; X)}.
\end{aligned}$$

By another application of Lemma 10.4.7,

$$\begin{aligned}
&\sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n U(e^{-i\epsilon\nu} 2^n t) x \right\|_{L^2(\Omega; X)} \\
&= \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n \int_0^\infty \xi(e^{-i\epsilon\nu} 2^n s t) f(A) \eta(sA) \psi(sA) x \frac{ds}{s} \right\|_{L^2(\Omega; X)} \\
&= \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n \int_0^\infty \xi(e^{-i\epsilon\nu} s t) f(A) \eta(2^{-n} sA) \psi(2^{-n} sA) x \frac{ds}{s} \right\|_{L^2(\Omega; X)} \\
&\leq \|\xi\|_{H^1(\Sigma_\sigma)} \sup_{t>0} \sup_{N \geq 1} \left\| \sum_{|m| \leq N} \varepsilon_m f(A) \eta(2^m tA) \psi(2^m tA) x \right\|_{L^2(\Omega; X)} \\
&\leq \|\xi\|_{H^1(\Sigma_\sigma)} \sup_{t>0} \mathcal{R}(\{f(A) \eta(2^m tA) : m \in \mathbb{Z}\}) \|x\|_{\psi, A}.
\end{aligned}$$

To estimate the R -bound uniformly in $t > 0$ we invoke Proposition 10.3.2, taking into account the estimate

$$\sup_{t>0} \int_{\partial\Sigma_\nu} |f(z)| |\eta(tz)| \left| \frac{dz}{z} \right| \leq c_{\phi, \psi}^{-1} \|f\|_{H^\infty(\Sigma_\sigma)} \|\xi\psi\|_{H^1(\Sigma_\sigma)}. \quad (10.34)$$

This gives the desired R -boundedness, with estimate

$$\mathcal{R}(\{f(A) \eta(2^m tA) : m \in \mathbb{Z}\}) \leq \frac{1}{\pi} M_{\sigma, A}^R c_{\phi, \psi}^{-1} \|f\|_{H^\infty(\Sigma_\sigma)} \|\xi\psi\|_{H^1(\Sigma_\sigma)}.$$

Putting together all the estimates we obtain part (1) with

$$C_{\phi, \psi} = \frac{1}{\pi^2} \|\phi\|_{H^1(\Sigma_\sigma)} \|\xi\|_{H^1(\Sigma_\sigma)} \|\xi\psi\|_{H^1(\Sigma_\sigma)} \|\xi\psi\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^{-2}. \quad (10.35)$$

□

As a first application we now show that the supremum over t in the definition of the discrete square function norm $\|\cdot\|_{\phi, A}$ can be eliminated under suitable assumptions on ϕ and A .

Proposition 10.4.8. *Let A be an R -sectorial operator, let $\omega_R(A) < \sigma < \pi$, and let $\phi \in H^1(\Sigma_\sigma) \cap H^2(\Sigma_\sigma)$ be a zero-free function with the property that the functions*

$$\psi_t(z) = \frac{\phi^2(tz)}{\phi(z)}, \quad t > 0,$$

satisfy

$$\sup_{t \in [1,2]} \|\psi_t\|_{H^1(\Sigma_\nu)} < \infty.$$

Then for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have an equivalence of norms

$$\|x\|_{\phi,A} \approx \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(2^n A) x \right\|^2 \right)^{1/2}$$

with constants independent of x . This proves the upper bound in (10.35). The lower bound (with constant 1) holds trivially.

It is easy to check that the condition on the ψ_t 's is satisfied for common choices such as $\phi(z) = z^\alpha e^{-z}$ and $\phi(z) = z^\alpha / (1+z)^\beta$ for $0 < \alpha < \beta$.

Proof. We apply Proposition 10.3.2, which gives us

$$\mathcal{R}(\{\psi_t(2^n A) : n \in \mathbb{Z}, t \in [1,2]\}) \leq \frac{1}{\pi} M_{\sigma,A}^R \sup_{t \in [1,2]} \|\psi_t\|_{H^1(\Sigma_\nu)} =: K.$$

For $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, by Proposition 10.4.6(3), with $C = (M_{\nu,A}^R)^2 C_{\phi,\xi}$ we obtain

$$\begin{aligned} \|x\|_{\phi,A} &\leq C \|x\|_{\phi^2,A} = C \sup_{t \in [1,2]} \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \phi^2(2^n t A) x \right\|^2 \right)^{1/2} \\ &= C \sup_{s \in [1,2]} \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \psi_t^2(2^n A) \phi(2^n A) x \right\|^2 \right)^{1/2} \\ &\leq CK \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(2^n A) x \right\|^2 \right)^{1/2}. \end{aligned}$$

□

Theorem 10.4.4 admits the following converse:

Theorem 10.4.9 (Square function bounds imply H^∞ -calculus). *Let A be an R -sectorial operator on a Banach space X . If, for some non-zero $\phi, \psi \in H^1(\Sigma_\sigma)$ with $\omega_R(A) < \sigma < \pi$, there exist constants $c_0, c_1 > 0$ such that*

$$\|x\| \leq c_0 \|x\|_{\phi,A}, \quad \text{and} \quad \|x\|_{\psi,A} \leq c_1 \|x\|, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A),$$

then A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus. In particular,

$$\omega_{H^\infty}(A) \leq \omega_R(A).$$

Proof. For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, from our assumption and Proposition 10.4.6 we infer that

$$\begin{aligned} \|f(A)x\| &\leq c_0 \|f(A)x\|_{\phi,A} \\ &\leq c_0 C_{\phi,\psi} (M_{\sigma,A}^R)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|x\|_{\psi,A} \\ &\leq c_0 c_1 C_{\phi,\psi} (M_{\sigma,A}^R)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|x\|, \end{aligned}$$

where $C_{\phi,\psi}$ is as in Proposition 10.4.6. This proves the result, with bound $M_{\sigma,A}^\infty \leq c_0 c_1 C_{\phi,\psi} (M_{\sigma,A}^R)^2$. $\square$

Corollary 10.4.10. *Let A be a densely defined sectorial operator with a bounded H^∞ -calculus on a Banach space X with the triangular contraction property. Then A is R -sectorial and*

$$\omega_{H^\infty}(A) = \omega_R(A) = \omega_\gamma(A).$$

Proof. By Theorem 10.3.4(2), A is R -sectorial and $\omega_R(A) \leq \omega_{H^\infty}(A)$. Furthermore, since the triangular contraction property implies finite cotype (Corollary 7.5.13), A is γ -sectorial with $\omega_R(A) = \omega_\gamma(A)$ by the equivalence of R -boundedness and γ -boundedness in such spaces (Theorem 8.1.3). The converse estimate $\omega_{H^\infty}(A) \leq \omega_R(A)$ has been proved in Theorem 10.4.9. $\square$

The following converse to Lemma 10.3.8 once more expresses the close connection between the H^∞ -calculus and unconditional convergence of the “parts” $\phi(2^n A)$.

Proposition 10.4.11. *Let A be a densely defined R -sectorial operator on a Banach space X and let $\omega_R(A) < \vartheta < \pi$. If, for some non-zero $\phi \in H^1(\Sigma_\vartheta)$,*

$$\sup_{t>0} \sup_{N \geq 1} \sup_{|\epsilon_{-N}|, \dots, |\epsilon_N|=1} \left\| \sum_{|n| \leq N} \epsilon_n \phi(2^n t A) \right\| < \infty,$$

then A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for any $\sigma \in (\omega_R(A), \pi)$.

Proof. By Remark 10.4.5, the square function estimates of Theorem 10.4.4 hold. It thus follows from Theorem 10.4.9 that A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for all $\omega_R(A) < \sigma < \pi$. $\square$

10.4.b Continuous square functions

We now turn to continuous analogues of the preceding discrete square function results. After some preliminary considerations on continuous square functions, our first aim is to obtain a continuous analogue of Theorem 10.4.4: Boundedness of the H^∞ -calculus implies boundedness of continuous square functions.

Definition 10.4.12 (Continuous square functions). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. The continuous square function norm of a vector $x \in X$ associated with a function $\phi \in H^1(\Sigma_\sigma)$ is defined by*

$$\|x\|_{\phi, A} := \left\| t \mapsto \phi(tA)x \right\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)}$$

whenever $t \mapsto \phi(tA)x$ belongs to $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$.

In order to state our results in their strongest possible form we will also need dual square functions. In order to introduce these we use the following terminology. For a measure space (S, Σ, μ) and a strongly μ -measurable function $g : S \rightarrow X^*$ we say that g belongs to $\gamma^*(S; X^*)$ if

$$\|g\|_{\gamma^*(S; X^*)} := \sup_{\substack{f \in \gamma(S; X) \\ \|f\|_{\gamma(S; X)} \leq 1}} \left| \int_S \langle f(s), g(s) \rangle d\mu(s) \right| < \infty,$$

the supremum being taken over all strongly measurable functions that are weakly square integrable and belong to $\gamma(S; X)$. By density it suffices to consider simple functions in the supremum.

Definition 10.4.13 (Continuous dual square functions). *Let A be a densely defined sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. The continuous dual square function norm of a vector $x^* \in X^*$ associated with a function $\phi \in H^1(\Sigma_\sigma)$ and A is defined by*

$$\|x^*\|_{\phi, A^*}^* := \left\| t \mapsto \phi(tA^*)x^* \right\|_{\gamma^*(\mathbb{R}_+, \frac{dt}{t}; X^*)}$$

whenever $t \mapsto \phi(tA^*)x^*$ belongs to $\gamma^*(\mathbb{R}_+, \frac{dt}{t}; X^*)$.

Typical choices in applications include

$$\begin{aligned} \phi(z) &= z^\alpha (1+z)^{-\beta}, & \phi(tA) &= (tA)^\alpha (I + tA)^{-\beta}; \\ \phi(z) &= z^\alpha \exp(-z), & \phi(tA) &= (tA)^\alpha \exp(-tA), \end{aligned}$$

where $0 < \alpha < \beta$. For the negative Laplace operator $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$, $1 < p < \infty$, the choice $\phi_k(z) = z^k \exp(-z)$ leads to a continuous square function norm equivalent to the classical Littlewood-Paley g -function norm of (10.30). To see this, recall that the radonifying norm in $L^p(\mathbb{R}^d)$ -space can be computed using Proposition 9.3.2. Then

$$\begin{aligned} \left\| \left(\int_0^\infty \left| t^k \frac{d^k}{dt^k} e^{-t\Delta} f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_p &\approx_p \left\| t \mapsto t^k \frac{d^k}{dt^k} e^{-t\Delta} f \right\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; L^p(\mathbb{R}^d))} \\ &= \left\| t \mapsto t^k (-\Delta)^k e^{-t\Delta} f \right\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; L^p(\mathbb{R}^d))} \\ &= \left\| t \mapsto \phi_k(tA) f \right\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; L^p(\mathbb{R}^d))}. \end{aligned}$$

We point out that the definition of continuous square functions is phrased in terms of membership of the space $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$. This space should be carefully distinguished from its larger variant $\gamma_\infty(\mathbb{R}_+, \frac{dt}{t}; X)$. In most applications X has finite cotype or is even a UMD space, and in such cases both spaces are equal, with identical norms, by Theorem 9.1.20. In the case of a general Banach space X , however, one must carefully distinguish between these spaces. One of our main tools in the analysis of continuous square functions is the γ -multiplier theorem, Theorem 9.5.1, which asserts that γ -bounded pointwise multipliers act boundedly from $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$ into $\gamma_\infty(\mathbb{R}_+, \frac{dt}{t}; X)$. This is only a minor inconvenience in the case of square functions, thanks to the following useful observation.

Lemma 10.4.14. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Then for all $\phi \in H^1(\Sigma_\sigma)$ and $x \in D(A) \cap R(A)$ the function $t \mapsto \phi(tA)x$ defines an element of $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$.*

Proof. Fix an $x \in D(A) \cap R(A)$, say $x = \zeta(A)y$ with $\zeta(z) = z(1+z)^{-2}$. With $\omega(A) < \nu < \sigma$, the Dunford calculus gives

$$\phi(tA)x = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \phi(tz) \zeta(z) R(z, A) y \, dz.$$

As it was already observed before (see the discussion around Theorem 10.4.4), Proposition H.2.4 implies that $t \mapsto \phi(tz)$ belongs to $L^2(\mathbb{R}_+, \frac{dt}{t})$. Then for each $x_0 \in X$ the function $t \mapsto \phi(tz) \otimes x_0$ belongs to $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$ and by Proposition 9.1.3 its norm is given by

$$\|t \mapsto \phi(tz) \otimes x_0\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} = \|t \mapsto \phi(tz)\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; X)} \|x_0\|.$$

Applying this with $x_0 = x_0(z) = R(z, A)y$, we may interpret the above Dunford integral as a Bochner integral in $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$. This shows that $t \mapsto \phi(tA)x \in \gamma(\mathbb{R}_+, \frac{dt}{t}; X)$ and

$$\begin{aligned} & \|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \\ & \leq \frac{1}{2\pi} \int_{\partial \Sigma_\nu} \|t \mapsto \phi(tz) \zeta(z) R(z, A)y\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} |dz| \\ & \leq \frac{M_{\nu, A}}{2\pi} \int_{\partial \Sigma_\nu} \|t \mapsto \phi(tz)\|_{L^2(\mathbb{R}_+, \frac{dt}{t})} |\zeta(z)| \|y\| \left| \frac{dz}{z} \right| \\ & \leq \frac{M_{\nu, A}}{\pi} \|\phi\|_{H^2(\Sigma_\sigma)} \|\zeta\|_{H^1(\Sigma_\sigma)} \|y\|. \end{aligned}$$

□

Our first result relating discrete and continuous square functions is as follows. A further comparison will be given in Proposition 10.4.20.

Proposition 10.4.15. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For $\phi \in H^1(\Sigma_\sigma)$ and $0 < \delta < \omega(A) - \sigma$ set $\phi_{\pm\delta}(z) = \phi(e^{\pm i\delta}z)$. Then,*

(1) *without any additional assumptions, for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have*

$$\|x\|_{\phi,A} \lesssim_\delta (\|x\|_{\phi_{-\delta},A} + \|x\|_{\phi_\delta,A});$$

(2) *if X has finite cotype, then for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have*

$$\|x\|_{\phi,A} \lesssim_\delta C_X \sup_{|\beta| < \delta} \|x\|_{\phi_\beta,A};$$

(3) *if A is densely defined, then for all $x^* \in \mathcal{D}(A^*) \cap \mathcal{R}(A^*)$ we have*

$$\|x^*\|_{\phi,A^*}^* \lesssim_\delta \sup_{|\beta| < \delta} \|x^*\|_{\phi_\beta,A^*}.$$

Proof. (1): We have

$$\begin{aligned} \|x\|_{\phi,A} &= \sup_{t \in [1,2]} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(2^n t A) x \right\|_{L^2(\Omega; X)} \\ &\leq \frac{1}{\|\gamma\|_1} \sup_{t \in [1,2]} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \gamma_n \phi(2^n t A) x \right\|_{L^2(\Omega; X)} \\ &\lesssim_\delta (\|x\|_{\phi_{-\delta},A} + \|x\|_{\phi_\delta,A}), \end{aligned}$$

where the last estimate follows from the sectorial version of Lemma 9.7.10.

(2): By the sectorial version of Proposition 9.7.20 we obtain

$$\begin{aligned} \|x\|_{\phi,A} &\lesssim_\delta \sup_{|\beta| < \delta} \sup_{t \in [1,2]} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \gamma_n \phi(t 2^n e^{i\beta} A) x \right\|_{L^2(\Omega; X)} \\ &\lesssim_\delta C_X \sup_{|\beta| < \delta} \sup_{t \in [1,2]} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(t 2^n e^{i\beta} A) x \right\|_{L^2(\Omega; X)} \\ &= C_X \sup_{|\beta| < \delta} \|x\|_{\phi_\beta,A}, \end{aligned}$$

where in the second step we applied the finite cotype of X to estimate the Gaussian sequence by a Rademacher sequence (see Corollary 7.2.10).

(3): By the sectorial version of Proposition 9.7.19,

$$\begin{aligned} \|x^*\|_{\phi,A^*}^* &\lesssim_\delta \sup_{|\beta| < \delta} \sup_{t \in [1,2]} \sup_{N \geq 1} \left\| \sum_{|n| \leq N} \varepsilon_n \phi(t 2^n e^{i\beta} A^*) x^* \right\|_{L^2(\Omega; X^*)} \\ &= \sup_{|\beta| < \delta} \|x^*\|_{\phi_\beta,A^*}^*. \end{aligned}$$

□

We can now prove the following continuous analogue of Theorem 10.4.4.

Theorem 10.4.16 (H^∞ -calculus implies square function bounds). *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus on a Banach space X with finite cotype, and let $\omega_{H^\infty}(A) < \sigma < \vartheta < \pi$. Let $\phi \in H^1(\Sigma_\vartheta)$.*

(1) *If X has finite cotype, then for all $x \in \overline{D(A) \cap R(A)}$ we have*

$$\|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C_{\vartheta, \sigma} C_X M_{\sigma, A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|.$$

(2) *If A is densely defined, then for all $x^* \in \overline{D(A^*) \cap R(A^*)}$,*

$$\|t \mapsto \phi(tA^*)x^*\|_{\gamma^*(\mathbb{R}_+, \frac{dt}{t}; X^*)} \leq C_{\vartheta, \sigma} M_{\sigma, A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x^*\|.$$

(3) *If A is densely defined and $\phi \neq 0$, then for all $x \in D(A) \cap R(A)$ we have*

$$\|x\| \leq C_{\vartheta, \sigma} C_\phi M_{\sigma, A}^\infty M_{\sigma, A} \|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)},$$

$$\text{where } C_\phi = \|\phi\|_{H^1(\Sigma_\vartheta)} \|\phi\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^{-2}.$$

In all three parts, membership of $t \mapsto \phi(tA)x$ of the respective spaces is part of the assertion.

Proof. (1): By density it suffices to consider $x \in D(A) \cap R(A)$. Let $\delta > 0$ be such that $\vartheta - \delta > \sigma$ (for instance $\delta = (\vartheta - \sigma)/2$). Since by Proposition 10.4.15(2),

$$\|x\|_{\phi, A} \lesssim_\delta C_X \sup_{|\beta| < \delta} \|x\|_{\phi_\beta, A},$$

it suffices to estimate $\|x\|_{\phi_\beta, A}$ uniformly for $|\beta| < \delta$. Since

$$\|\phi_\beta\|_{H^1(\Sigma_{\vartheta-\delta})} \leq \|\phi\|_{H^1(\Sigma_\vartheta)},$$

it follows from Theorem 10.4.4(1) that

$$\|x\|_{\phi_\beta, A} \lesssim_{\sigma, \vartheta} M_{\sigma, A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|$$

with a constant depending on σ, ϑ but independent of β .

(2): This can be proved in the same way by using Proposition 10.4.15(3) and Theorem 10.4.4(2) instead.

(3): Since we take $x \in D(A) \cap R(A)$, Lemma 10.4.14 guarantees that $\phi(tA)x \in \gamma(\mathbb{R}_+, \frac{dt}{t}; X)$. Let $\psi \in H^1(\Sigma_\vartheta)$ be given by $\psi(z) = c^{-1}\phi(\bar{z})$ with $c = \int_0^\infty |\phi(t)|^2 \frac{dt}{t}$. For all $x^* \in D(A^*) \cap R(A^*)$, from the reproducing formula of Proposition 10.2.5 we obtain

$$|\langle x, x^* \rangle| = \left| \int_0^\infty \langle \phi(tA)\psi(tA)x, x^* \rangle \frac{dt}{t} \right|$$

$$\leq \|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \|t \mapsto \psi(tA)^* x^*\|_{\gamma^*(\mathbb{R}_+, \frac{dt}{t}; X^*)}.$$

By (2),

$$\|\psi(tA)^* x^*\|_{\gamma^*(\mathbb{R}_+, \frac{dt}{t}; X^*)} \leq C_{\vartheta, \sigma} M_{\sigma, A}^\infty \|\phi\|_{H^1(\Sigma_\vartheta)} \|x^*\|.$$

Therefore,

$$|\langle x, x^* \rangle| \leq C_{\vartheta, \sigma} M_{\sigma, A}^\infty c^{-1} \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|_{\phi, A} \|x^*\|.$$

Taking the supremum over all $x^* \in D(A^*) \cap R(A^*)$ of norm ≤ 1 , from Lemma 10.2.19 we infer

$$\|x\| \leq 2C_{\vartheta, \sigma} M_{\sigma, A} M_{\sigma, A}^\infty c^{-1} \|\phi\|_{H^1(\Sigma_\vartheta)} \|x\|_{\phi, A}.$$

□

Our next aim is to prove that boundedness of continuous square functions implies the boundedness of the H^∞ -calculus. Here we will follow the same strategy as in the discrete setting. The following result is the continuous analogue of Proposition 10.4.6.

Let $M_{\sigma, A}^\gamma$ denote the γ -bound of the set $\{zR(z, A) : z \in \overline{\mathbb{C} \setminus \Sigma_\nu}\}$.

Proposition 10.4.17 (Equivalence of continuous square functions).

Let A be a γ -sectorial operator on X and let $\omega_\gamma(A) < \nu < \sigma < \pi$. Fix arbitrary $\phi, \psi \in H^1(\Sigma_\sigma)$, with $\psi \neq 0$. Then the following assertions hold:

(1) for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $x \in D(A) \cap R(A)$ we have

$$\|f(A)\phi(\cdot A)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C_{\phi, \psi} (M_{\sigma, A}^\gamma)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|\psi(\cdot A)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)}.$$

(2) if both ϕ and ψ are non-zero, then for all $x \in D(A) \cap R(A)$ we have an equivalence of norms

$$\|\phi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} \approx \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)}$$

with constants independent of x .

In the proof we use the following lemma.

Lemma 10.4.18. For any $h \in L^1(\mathbb{R}_+, \frac{dt}{t})$, the operator S_h on $L^2(\mathbb{R}_+, \frac{dt}{t}) \otimes X$ defined by

$$S_h(u \otimes x)(s) := \left(\int_0^\infty h(st)u(t) \frac{dt}{t} \right) \otimes x, \quad s > 0,$$

extends uniquely to a bounded operator on $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$ of norm $\|S_h\| \leq \|h\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}$.

Proof. By Theorem 9.6.1 it suffices to show that $u \mapsto \int_0^\infty h(t \cdot) u(t) \frac{dt}{t}$ defines a bounded operator on $L^2(\mathbb{R}_+, \frac{dt}{t})$. Let $c = \|h\|_{L^1(\mathbb{R}_+, \frac{dt}{t})}$. By Jensen's inequality and Fubini's theorem, for all $u \in L^2(\mathbb{R}_+, \frac{dt}{t})$ we have

$$\int_0^\infty \left| \int_0^\infty h(st) u(t) \frac{dt}{ct} \right|^2 \frac{ds}{s} \leq \int_0^\infty \int_0^\infty |h(st)| |u(t)|^2 \frac{dt}{ct} \frac{ds}{s} = \int_0^\infty |u(t)|^2 \frac{dt}{t}.$$

This gives the result. $\square$

Proof of Proposition 10.4.17. As in the discrete case we only need to prove assertion (1). Since $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, all square functions are well defined by Lemma 10.4.14. It was shown in the proof of Proposition 10.4.6 (see (10.33)), using the same notation introduced there, that

$$\begin{aligned} f(A)\phi(sA) &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \phi(sz) R(z, A) U(z) dz \\ &= \frac{1}{2\pi i} \sum_{\epsilon=\pm 1} \int_0^\infty \epsilon \phi(st e^{-i\epsilon\nu}) t e^{-i\epsilon\nu} R(t e^{-i\epsilon\nu}, A) U(t e^{-i\epsilon\nu}) \frac{dt}{t}. \end{aligned}$$

Using Lemma 10.4.18, the γ -multiplier theorem (Theorem 9.5.1 applied in combination with Lemma 10.4.14), the γ -sectoriality of A , and Lemma 10.4.18 once again, we obtain

$$\begin{aligned} \|t \mapsto f(A)\phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} &\leq \frac{1}{\pi} \|\phi\|_{H^1(\Sigma_\sigma)} \sup_{\epsilon \in \{-1, 1\}} \|t \mapsto t R(t e^{i\epsilon\nu}, A) U(t e^{i\epsilon\nu}) x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\leq \frac{1}{\pi} M_{\nu, A}^\gamma \|\phi\|_{H^1(\Sigma_\sigma)} \sup_{\epsilon \in \{-1, 1\}} \|t \mapsto U(t e^{i\epsilon\nu}) x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\leq \frac{1}{\pi} M_{\nu, A}^\gamma \|\phi\|_{H^1(\Sigma_\sigma)} \|\xi\|_{H^1(\Sigma_\sigma)} \|t \mapsto f(A)\eta(tA)\psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)}. \end{aligned}$$

In view of the γ -multiplier theorem, it remains to be shown that $\{f(A)\eta(tA) : t > 0\}$ is γ -bounded. But this follows from the analogue of Proposition 10.3.2 for γ -boundedness, taking into account the estimate (10.34). The γ -multiplier theorem applied in combination with Lemma 10.4.14, now gives

$$\begin{aligned} \|t \mapsto f(A)\eta(tA)\psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} &\leq \frac{1}{\pi} (M_{\nu, A}^\gamma)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|\phi\|_{H^1(\Sigma_\sigma)} \|\xi\psi\|_{H^1(\Sigma_\sigma)} \|t \mapsto \psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)}. \end{aligned}$$

After letting $\nu \uparrow \sigma$, this proves (1) with constant $C_{\phi, \psi}$ as in (10.35). $\square$

The next result is a continuous time version of Theorem 10.4.9 and a converse to Theorem 10.4.16.

Theorem 10.4.19 (Square function bounds imply H^∞ -calculus). *Let A be a γ -sectorial operator on a Banach space X . If, for some non-zero $\phi, \psi \in H^1(\Sigma_\sigma)$ with $\omega_\gamma(A) < \sigma < \pi$, for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have*

$$\|x\| \leq c_0 \|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \text{ and } \|t \mapsto \psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq c_1 \|x\|,$$

then A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus. In particular,

$$\omega_{H^\infty}(A) \leq \omega_\gamma(A).$$

Recall from Corollary 10.4.10 that $\omega_{H^\infty}(A) = \omega_\gamma(A)$ if we make the additional assumption that X has the triangular contraction property.

Proof. Fix an arbitrary $\omega_\gamma(A) < \nu < \sigma$. For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $x \in \overline{\mathcal{R}(A)}$, from Proposition 10.4.17 we infer that

$$\begin{aligned} \|f(A)x\| &\leq c_0 \|t \mapsto f(A)\phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\leq c_0 C_{\phi, \psi}(M_{\sigma, A}^\gamma)^2 \|f\|_{H^\infty(\Sigma_\sigma)} \|f\|_\infty \|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\leq c_0 c_1 C_{\phi, \psi}(M_{\sigma, A}^\gamma)^2 \|f\|_\infty \|x\|, \end{aligned}$$

where $C_{\phi, \psi}$ is as in Proposition 10.4.17. This proves the result, with bound $M_{\nu, A}^\infty \leq c_0 c_1 C_{\phi, \psi}(M_{\sigma, A}^\gamma)^2$. $\square$

In Proposition 10.4.15 we have compared discrete and continuous square functions at different angles. As a final result in this section we give conditions under which discrete and continuous square functions are actually equivalent for a fixed angle.

Proposition 10.4.20. *Let A be an R -sectorial operator on a Banach space X with finite cotype. Let $\omega(A) < \sigma < \vartheta < \pi$. For all $\phi \in H^1(\Sigma_\vartheta)$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have*

$$\|x\|_{\phi, A} \approx \|x\|_{\phi, A}$$

with a constant independent of x .

Proof. By Proposition 10.4.15(2) and Proposition 10.4.6, we have

$$\|x\|_{\phi, A} \lesssim_\delta C_X \sup_{|\beta| < \delta} \|x\|_{\phi_\beta, A} \lesssim_{\delta, \phi} C_X (M_{\nu, A}^R)^2 C_{\phi, \delta} \|x\|_{\phi, A}.$$

The converse estimate follows in the same way by instead applying Propositions 10.4.15(1) and 10.4.17. $\square$

Sectorial operators on Hilbert spaces revisited

In Hilbert spaces, square function estimates take a particularly simple form, since by Proposition 9.2.9 we have

$$L^2(\mathbb{R}_+, \frac{dt}{t}; H) \simeq \gamma(\mathbb{R}_+, \frac{dt}{t}; H)$$

isometrically. For this reason, they are sometimes referred to as *quadratic estimates*. Before coming to the main results, we also recall Proposition 10.1.9, which implies that sectorial operators in Hilbert spaces A are densely defined and satisfy $\overline{D(A)} \cap \overline{R(A)} = \overline{R(A)}$.

We have the following fundamental characterisation of sectorial Hilbert space operators with a bounded H^∞ -calculus of angle less than $\frac{1}{2}\pi$.

Theorem 10.4.21 (McIntosh). *For a sectorial operator A on a Hilbert space H the following assertions are equivalent:*

- (1) A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for some $\sigma \in (\omega(A), \pi)$;
- (2) A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus for all $\sigma \in (\omega(A), \pi)$;
- (3) for some $\sigma \in (\omega(A), \pi)$ and some non-zero $\phi, \psi \in H^1(\Sigma_\sigma)$ there exist constants $c_0 \geq 0$ and $c_1 \geq 0$ such that

$$\|x\| \leq c_0 \|t \mapsto \phi(tA)x\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} \text{ and } \|t \mapsto \psi(tA)x\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} \leq c_1 \|x\|$$

for all $x \in \overline{R(A)}$;

- (4) for all $\sigma \in (\omega(A), \pi)$ and all non-zero $\phi, \psi \in H^1(\Sigma_\sigma)$ there exist constants $c_0 \geq 0$ and $c_1 \geq 0$ such that

$$\|x\| \leq c_0 \|t \mapsto \phi(tA)x\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} \text{ and } \|t \mapsto \psi(tA)x\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} \leq c_1 \|x\|$$

for all $x \in \overline{R(A)}$.

Moreover, we have equality of the bounds

$$\omega_{H^\infty}(A) = \omega_R(A) = \omega_\gamma(A) = \omega(A).$$

Proof. The equality $\omega_R(A) = \omega_\gamma(A) = \omega(A)$ is an immediate consequence of the fact that the notions of R -sectoriality, γ -sectoriality and sectoriality coincide in Hilbert spaces by Theorem 8.1.3.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4): These implications follow from Theorem 10.4.16.

(3) $\Rightarrow$ (2): This follows from Theorem 10.4.19 which also gives the inequality $\omega_{H^\infty}(A) \leq \omega(A)$; the converse inequality holds trivially.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3): These implications are trivial. $\square$

If $-A$ is the generator of a C_0 -semigroup of contractions on Hilbert space, then A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$ by Theorem 10.2.24. The next result shows that in the case $\omega(A) < \frac{1}{2}\pi$, this actually gives a characterisation of the boundedness of the H^∞ -calculus.

Theorem 10.4.22 (Le Merdy). *A be a sectorial operator on a Hilbert space H with dense range and which satisfies $\omega(A) < \frac{1}{2}\pi$. Then the following assertions are equivalent:*

- (1) *A has a bounded H^∞ -calculus;*
- (2) *$-A$ generates C_0 -contraction semigroup with respect to some equivalent Hilbertian norm on H .*

Proof. (1) $\Rightarrow$ (2): By Theorem 10.4.21 we have $\omega_{H^\infty}(A) = \omega(A) < \frac{1}{2}\pi$. Furthermore, Lemma 10.2.8, for the function $\varphi(z) = z \exp(-z)$ we have $\varphi(tA) = tAS(t)$, $t \geq 0$, where $(S(t))_{t \geq 0}$ denotes the bounded analytic C_0 -semigroup generated by $-A$ (see Theorem G.5.2). By Theorem 10.4.16,

$$\|x\| := \|t \mapsto \varphi(tA)x\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} = \|t \mapsto t^{1/2}AS(t)x\|_{L^2(\mathbb{R}_+; H)}$$

defines an equivalent Hilbertian norm on H , and the estimate

$$\|S(s)x\| = \|t \mapsto t^{1/2}AS(s+t)x\|_{L^2(\mathbb{R}_+; H)} \leq \|t \mapsto t^{1/2}AS(t)x\|_{L^2(\mathbb{R}_+; H)} = \|x\|$$

shows that $(S(t))_{t \geq 0}$ is contractive in this norm.

(2) $\Rightarrow$ (1): This follows from Theorem 10.2.24. $\square$

Square functions and H^∞ -calculus in L^p -spaces

Theorem 10.4.23. *Let $(S, \mathcal{A}, \mu)$ be a σ -finite measure space and $p \in (1, \infty)$. Let A be a sectorial operator on $L^p(S)$. Consider the following assertions for $\sigma, \vartheta \in (\omega(A), \pi)$:*

- (1) *A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus;*
- (2) *A is R -sectorial with $\omega_R(A) < \vartheta$ and for some (equivalently, all) non-zero $\phi \in H^1(\Sigma_\vartheta)$ there exist constants C_0 and C_1 such that*

$$C_0^{-1}\|x\|_p \leq \left\| \left(\int_0^\infty |\phi(tA)x|^2 dt \right)^{1/2} \right\|_p \leq C_1\|x\|_p$$

for all $x \in \overline{R(A)}$.

Then (1) $\Rightarrow$ (2) holds if $\vartheta > \sigma$, and (2) $\Rightarrow$ (1) holds if $\sigma \geq \vartheta$.

The constants in (2) can be taken as $C_0 = \|\phi\|_{L^2(\mathbb{R}_+, \frac{dt}{t})}^{-2} C$ and $C_1 = C_p C$, where $C := C_{\vartheta, \sigma} M_{\sigma, A}^\infty M_{\sigma, A} \|\phi\|_{H^1(\Sigma_\vartheta)}$. The theorem remains true if $L^p(S)$ is replaced by any reflexive Banach function space with finite cotype.

Proof. Since $L^p(S)$ is reflexive, $D(A)$ is dense by Proposition 10.1.9. Since $L^p(S)$ has the triangular contraction principle and finite cotype, the implication (1) $\Rightarrow$ (2) for $\vartheta > \sigma$ is immediate from Theorems 10.3.4 (R -sectoriality) and 9.3.8 and 10.4.16 (square function estimates). The implication (2) $\Rightarrow$ (1) for $\sigma \geq \vartheta$ follows from Theorem 10.4.19, recalling that R -sectoriality implies γ -sectoriality. $\square$

Similarly, Theorem 10.3.4 and Corollary 10.4.10 imply the following result.

Theorem 10.4.24. *Let $(S, \mathcal{A}, \mu)$ be a σ -finite measure space and $p \in (1, \infty)$. Let A be a sectorial operator with dense range and with a bounded $H^\infty(\Sigma_\sigma)$ -calculus on $L^p(S)$. Then A is R -sectorial, $\omega_R(A) = \omega_{H^\infty}(A)$ and the set*

$$\{f(A) : f \in H^\infty(\Sigma_\sigma), \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$$

is R -bounded, with R -bound at most $C_{\eta, \sigma}(M_{\eta, A}^\infty)^2 C_p$, for any $\omega(A) < \eta < \sigma$.

10.4.c The quadratic H^∞ -calculus

In the preceding subsection we have characterised, under suitable assumptions on A and X , the boundedness of the H^∞ -calculus of a sectorial operator A on X in terms of continuous square function estimates of the form

$$\|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \approx \|x\|$$

under the assumption $\phi \in H^1(\Sigma)$ for a suitable sector Σ . By restriction, such functions belong to $H^\infty(\Sigma')$, and hence to $H^p(\Sigma')$ for all $1 < p < \infty$, for any sector Σ' properly contained in Σ (see Proposition H.2.4). In particular they belong to $H^2(\Sigma')$, and the quadratic nature of square functions prompts the natural question whether the above equivalence of norms extends to arbitrary functions $\phi \in H^2(\Sigma)$. The affirmative answer to this question will be deduced from another result, which is of considerable interest in itself: the mapping $\phi \mapsto \phi(A)$, viewed as a bounded mapping

$$H^\infty(\Sigma) \rightarrow \mathcal{L}(X),$$

extends to a bounded linear mapping

$$H^\infty(\Sigma; H^*) \rightarrow \mathcal{L}(X, \gamma(H, X))$$

for any Hilbert space H , provided X has finite cotype. Taking $H = L^2(\mathbb{R}_+, \frac{dt}{t})$ will answer the question above (see Theorem 10.4.27).

Theorem 10.4.25 (Haak and Haase, Le Merdy). *Let A be a sectorial operator on a Banach space X with finite cotype q , with dense range and dense domain, and assume that A has a bounded $H^\infty(\Sigma_\sigma)$ -functional calculus. Let H be a Hilbert space, let $\sigma < \vartheta < \pi$, and define, for $F \in H^\infty(\Sigma_\vartheta; H^*)$, the operator $F(A) : X \rightarrow \mathcal{L}(H, X)$ by*

$$[F(A)x]h := \langle h, F \rangle(A)x, \quad x \in X, \quad h \in H.$$

Then $F(A)$ maps X into $\gamma(H, X)$ boundedly and

$$\|F(A)\|_{\mathcal{L}(X, \gamma(H, X))} \leq C_{q, X} C_{\sigma, \vartheta} (M_{\sigma, A}^\infty)^2 \|F\|_{H^\infty(\Sigma_\sigma; H^*)},$$

where $C_{q, X} = 2c_{q, X} q^{1/q} \|\gamma\|_{2q}$, where γ is a standard Gaussian variable.

We start with a preliminary result on sequences of holomorphic functions which is an analogue of Lemma 10.3.8(1).

Lemma 10.4.26. *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -functional calculus on a Banach space X . Let $f = (f_n)_{n \geq 1} \in H^\infty(\Sigma_\sigma; \ell^1)$. Then for all finite sets $I \subseteq \mathbb{N}_{>0}$ and scalars $|\alpha_n| \leq 1$ ($n \in I$),*

$$\left\| \sum_{n \in I} \alpha_n f_n(A)x \right\| \leq M_{\sigma, A}^\infty \|f\|_{H^\infty(\Sigma_\sigma; \ell^1)} \|x\|, \quad x \in \overline{D(A)} \cap R(A).$$

Proof. Let $I \subseteq \mathbb{N}$ be finite and let $\phi_I = \sum_{n \in I} \alpha_n f_n$. Then $\phi_I \in H^\infty(\Sigma_\sigma)$ and $\|\phi_I\|_{H^\infty(\Sigma_\sigma)} \leq \|f\|_{H^\infty(\Sigma_\sigma; \ell^1)}$. Therefore, for all $x \in \overline{D(A)} \cap R(A)$,

$$\left\| \sum_{n \in I} \alpha_n f_n(A)x \right\| = \|\phi_I(A)x\| \leq M_{\sigma, A}^\infty \|f\|_{H^\infty(\Sigma_\sigma; \ell^1)} \|x\|$$

by the boundedness of the H^∞ -calculus. $\square$

Lemma 10.4.26 is the only place where the boundedness of the functional calculus is used.

Proof of Theorem 10.4.25. We may assume that $\|F\|_{H^\infty(\Sigma_\sigma; H^*)} = 1$. By Theorem H.3.1, F admits a representation of the form

$$F(z) = \sum_{n \geq 1} f_n(z) g_n(z) h_n^*, \quad z \in \Sigma_\sigma,$$

where $\|(f_n)_{n \geq 1}\|_{H^\infty(\Sigma_\sigma; \ell^1)} \leq 1$, $\|(g_n)_{n \geq 1}\|_{H^\infty(\Sigma_\sigma; \ell^1)} \leq 1$, and $\|h_n^*\| \leq C_{\sigma, \vartheta}$. Moreover, by the properties of the functional calculus (see Theorem 10.2.13), for all $h \in H$ and $x \in X$ we have

$$[(h, F)](A)x = \sum_{n \geq 1} \langle h, h_n^* \rangle f_n(A) g_n(A) x.$$

Let $(h_j)_{j \in J}$ be a finite orthonormal system in H and let $(\gamma_j)_{j \geq 1}$ be a Gaussian sequence on a probability space $(\Omega, \mathbb{P})$. For $n \geq 1$ put

$$\hat{\gamma}_n := \sum_{j \in J} \gamma_j \langle h_j, h_n^* \rangle.$$

Then each $\hat{\gamma}_n$ is a Gaussian random variable and for all $r \in [1, \infty)$ we have

$$\|\hat{\gamma}_n\|_r = \|\gamma\|_r \left(\sum_{j \in J} |\langle h_j, h_n^* \rangle|^2 \right)^{1/2} \leq \|\gamma\|_r \|h_n^*\| \leq C_{\sigma, \vartheta} \|\gamma\|_r. \quad (10.36)$$

Using the identity $[F(A)x]h_j = [\langle h_j, F \rangle](A)x$ and the representation formula for F , we obtain

$$\begin{aligned} \left\| \sum_{j \in J} \gamma_j [F(A)x] h_j \right\|_{L^2(\Omega; X)} &= \left\| \sum_{n \geq 1} \hat{\gamma}_n f_n(A) g_n(A) x \right\|_{L^2(\Omega; X)} \\ &\leq \liminf_{N \rightarrow \infty} \left\| \sum_{n=1}^N \hat{\gamma}_n f_n(A) g_n(A) x \right\|_{L^2(\Omega; X)}. \end{aligned}$$

Let $(\varepsilon'_n)_{n \geq 1}$ be a Rademacher sequence on a distinct probability space $(\Omega', \mathbb{P}')$. By randomisation and Lemma 10.4.26 applied pointwise in Ω , we have

$$\begin{aligned} \left\| \sum_{n=1}^N \hat{\gamma}_n f_n(A) g_n(A) x \right\|_{L^2(\Omega; X)} &\leq \left\| \sum_{m=1}^N \varepsilon'_m f_m(A) \sum_{n=1}^N \hat{\gamma}_n \varepsilon'_n g_n(A) x \right\|_{L^2(\Omega \times \Omega'; X)} \\ &\leq M_{\sigma, A}^\infty \left\| \sum_{n=1}^N \hat{\gamma}_n \varepsilon'_n g_n(A) x \right\|_{L^2(\Omega \times \Omega'; X)}. \end{aligned}$$

Since X has cotype $q < \infty$, from Theorem 7.2.6(2) and (10.36) we deduce that

$$\left\| \sum_{n=1}^N \hat{\gamma}_n \varepsilon'_n g_n(A) x \right\|_{L^2(\Omega \times \Omega'; X)} \leq C_{q, X} C_{\sigma, \vartheta} \left\| \sum_{n=1}^N \varepsilon'_n g_n(A) x \right\|_{L^q(\Omega'; X)},$$

where $C_{q, X} = 2c_{q, X} q^{1/q} \|\gamma\|_{2q}$. By Lemma 10.4.26 applied pointwise in Ω' , we have $\left\| \sum_{n=1}^N \varepsilon'_n g_n(A) x \right\| \leq M_{\sigma, A}^\infty \|x\|$. Combining all estimates we obtain the estimate

$$\|F(A)x\|_{\gamma_\infty(H, X)} \leq \sup_h \left\| \sum_{j \in J} \gamma_j [F(A)x] h_j \right\|_{L^2(\Omega; X)} \leq C_{q, X} C_{\sigma, \vartheta} (M_{\sigma, A}^\infty)^2 \|x\|,$$

where the supremum extends over all finite orthonormal systems $h = (h_j)_{j \in J}$ in H . This proves the required result, since by Theorem 9.1.20 we have $\gamma_\infty(H, X) = \gamma(H, X)$, for the finite cotype assumption implies that X does not contain an isomorphic copy of c_0 . $\square$

Next we consider the special case $H = L^2(\mathbb{R}_+, \frac{dt}{t})$ and obtain the announced extension of Theorem 10.4.161 to functions $\phi \in H^2(\Sigma_\vartheta)$.

Theorem 10.4.27 (γ -Square functions). *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus on a Banach space X with finite cotype q . Let $0 < \sigma < \vartheta < \pi$. For all $\phi \in H^2(\Sigma_\vartheta)$ and all $x \in \overline{D(A)} \cap \mathcal{R}(A)$ the function $t \mapsto \phi(tA)x$ belongs to $\gamma(\mathbb{R}_+, \frac{dt}{t}, X)$ and*

$$\|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}, X)} \leq C_{q, X} C_{\sigma, \vartheta} (M_{\sigma, A}^\infty)^2 \|\phi\|_{H^2(\Sigma_\vartheta)} \|x\|$$

where the constants $C_{q, X}$ and $C_{\sigma, \vartheta}$ are as in Theorem 10.4.25.

Proof. Let $\sigma < \vartheta' < \vartheta$. The assumption on ϕ implies that $F(z) := [t \mapsto \phi(tz)]$ defines an element of $H^\infty(\Sigma_\vartheta; L^2(\mathbb{R}_+, \frac{dt}{t}))$ of norm $\|F\|_{H^\infty(\Sigma_\vartheta; L^2(\mathbb{R}_+, \frac{dt}{t}))} = \|\phi\|_{H^2(\Sigma_\vartheta)}$. Using Lemma 10.2.16, for $h \in L^2(\mathbb{R}_+, \frac{dt}{t})$ we have

$$[F(A)x]h = \int_0^\infty h(t)\phi(tA)x \frac{dt}{t}.$$

This shows that in $\mathcal{L}(L^2(\mathbb{R}_+, \frac{dt}{t}), X)$ we have $F(A)x = [t \mapsto \phi(tA)x]$. Now by Theorem 10.4.25, for all $x \in \overline{D(A) \cap R(A)}$ we obtain that $t \mapsto \phi(tA)x$ belongs to $\gamma(\mathbb{R}_+, \frac{dt}{t}, X)$ and gives the estimate

$$\|t \mapsto \phi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}, X)} \leq C_{q,X} C_{\sigma,\vartheta} (M_{\sigma,A}^\infty)^2 \|\phi\|_{H^2(\Sigma_\vartheta)} \|x\|.$$

□

10.4.d Le Merdy's theorem on the angle of the H^∞ -calculus

In the previous section it has been shown that if A is densely defined and has a bounded H^∞ -calculus on a Banach space with the triangular contraction property, then A is R -sectorial and $\omega_R(A) = \omega_{H^\infty}(A)$. In many applications one knows that $\omega(A) < \frac{1}{2}\pi$, for instance when A is given by a sectorial form of when $-A$ generates a bounded analytic C_0 -semigroup (see Appendix G for the precise statements). In such situations it is natural to ask whether one also has $\omega_R(A) = \omega_{H^\infty}(A) < \frac{1}{2}\pi$. Since there are many interesting holomorphic functions which are bounded on every proper subsector of $\mathbb{C}_+$, such as $z^\alpha \exp(-z)$ with $\alpha > 0$, knowing that the angles are strictly smaller than $\frac{1}{2}\pi$ is quite a useful piece of information. In this section we present a theorem due to Le Merdy describing a situation where this happens. It prominently features the functions

$$\varphi_\alpha(z) := z^\alpha \exp(-z), \quad z \in \mathbb{C}_+,$$

where $\alpha > 0$. If $\omega(A) < \frac{1}{2}\pi$, then the operators $\varphi_\alpha(tA)$ are well defined for all $t > 0$ by the Dunford calculus, and Lemma 10.4.14 shows that if all $x \in D(A) \cap R(A)$, then the function $t \mapsto \varphi_\alpha(tA)x$ belongs to $\gamma(\mathbb{R}_+, \frac{dt}{t}; X)$. The theorem is concerned with what happens if a two-sided square function estimate holds:

Theorem 10.4.28 (Le Merdy). *Let A be a densely defined sectorial operator with dense range, and which satisfies $\omega(A) < \frac{1}{2}\pi$, on a Banach space X with Pisier's contraction property. Suppose there is a constant $C > 0$ such that the following two-sided estimate holds for all $x \in D(A) \cap R(A)$:*

$$C^{-1}\|x\| \leq \|t \mapsto \varphi_{1/2}(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C\|x\|. \quad (10.37)$$

Then A is R -sectorial and has a bounded H^∞ -calculus, and we have

$$\omega_R(A) = \omega_{H^\infty}(A) < \frac{1}{2}\pi.$$

All the steps in the proof rely on explicit quantitative estimates involving the relevant constants, namely, the angle of sectoriality $\omega(A)$, the sectoriality constants $M_{\sigma,A}$ for $\omega(A) < \sigma < \frac{1}{2}\pi$, the constants associated with Pisier's contraction property, and the constants in (10.37). As a consequence it is possible to give an explicit estimate for the difference $\frac{1}{2}\pi - \omega_R(A)$. Since the resulting expression is complicated and not likely to provide any new insights, we leave this to the interested reader.

Pisier's contraction property will enter through Corollary 9.4.11, which implies that if X has this property, then we have a natural isomorphism of Banach spaces

$$\gamma(T \times S; X) \simeq \gamma(T; \gamma(S; X)) \quad (10.38)$$

whenever (S, Σ, μ) and $(T, \mathcal{B}, \nu)$ are σ -finite measure spaces.

We begin with a simple extension result.

Lemma 10.4.29. *Let X, Y be Banach spaces, $(S, \mathcal{A}, \mu)$ and $(T, \mathcal{B}, \nu)$ measure spaces, and let $\Psi: T \rightarrow \mathcal{L}(X, Y)$ a function such that for every $x \in X$ the function $t \mapsto \Psi(t)x$ belongs to $\gamma(T, Y)$ and*

$$C^{-1}\|x\| \leq \|t \mapsto \Psi(t)x\|_{\gamma(T; X)} \leq C\|x\|.$$

If $f \in \gamma(S; X)$, then $t \mapsto [s \mapsto \Psi(t)f(s)]$ belongs to $\gamma(T; \gamma(S; Y))$ and

$$C^{-1}\|f\|_{\gamma(S; X)} \leq \|\Psi f\|_{\gamma(T; \gamma(S; Y))} \leq C\|f\|_{\gamma(S; X)}.$$

Proof. First we check that the mapping $t \mapsto \Psi(t)f(\cdot)$ is strongly ν -measurable as a function from T to $\gamma(S, Y)$. By Example 9.1.16 it suffices to check that $t \mapsto \Psi(t)\mathbb{I}_f h$ is measurable for all $h \in L^2(S)$, and this is clear from the assumptions.

The next step is to show that the mapping $t \mapsto [s \mapsto \Psi(t)f(s)]$ defines an element of $\gamma(T, \gamma(S, Y))$ and satisfies the required norm estimate for any $f \in \gamma(S; X)$. Recall from Proposition 9.4.9 that this space isometrically coincides with $\gamma(S, \gamma(T, Y))$.

First consider a μ -simple function $f = \sum_{n=1}^N \frac{1}{\sqrt{|A_n|}} \mathbf{1}_{A_n} \otimes x_n$ with disjoint measurable sets $A_n \subseteq S$ of finite positive μ -measure. Then $t \mapsto \Psi(t)f(\cdot) = \sum_{n=1}^N \frac{1}{\sqrt{|A_n|}} \mathbf{1}_{A_n}(\cdot) \Psi(t)x_n$ is a μ -simple function with values in $\gamma(T; Y)$ and therefore, by Proposition 9.1.3, it belongs to $\gamma(S; \gamma(T; Y))$ and its norm is given by

$$\begin{aligned} \|t \mapsto [s \mapsto \Psi(t)f(s)]\|_{\gamma(T, \gamma(S, Y))}^2 &= \|s \mapsto [t \mapsto \Psi(t)f(s)]\|_{\gamma(S, \gamma(T, Y))}^2 \\ &= \mathbb{E} \left\| \sum_{n=1}^N \gamma_n [t \mapsto \Psi(t)x_n] \right\|_{\gamma(T; Y)}^2 \\ &\leq C^2 \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2 = C^2 \|f\|_{\gamma(S; X)}^2, \end{aligned}$$

where in the penultimate step we used the assumption on Ψ . By a density argument (see Proposition 9.2.5) the result extends to general $f \in \gamma(S; X)$.

The lower estimate also follows from

$$C^2 \mathbb{E} \left\| \sum_{n=1}^N \gamma_n [t \mapsto \Psi(t)x_n] \right\|_{\gamma(T; Y)}^2 \geq \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|_X^2 = \|f\|_{\gamma(S; X)}^2,$$

using the assumed lower bound for Ψ . $\square$

The next lemma takes advantage of the specific properties of the functions $\varphi_\alpha(z) = z^\alpha \exp(-z)$.

Lemma 10.4.30. *Let X be a Banach space with Pisier's contraction property. Let $\alpha, \beta > 0$ and assume that there constants $C_{0,A}$ and $C_{1,A}$ such that*

$$\begin{aligned} C_{0,A}^{-1} \|x\| &\leq \|t \mapsto \varphi_\alpha(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C_{0,A} \|x\|, \\ C_{1,A}^{-1} \|x\| &\leq \|t \mapsto \varphi_\beta(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C_{1,A} \|x\|, \end{aligned}$$

for all $x \in D(A) \cap R(A)$. Then for all $x \in D(A) \cap R(A)$,

$$C^{-1} \|x\| \leq \|t \mapsto \varphi_{\alpha+\beta}(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C \|x\|,$$

where $C = C_X C_{\alpha,\beta} C_{0,A} C_{1,A}$ for some constants $C_X, C_{\alpha,\beta} > 0$.

Proof. By the multiplicativity of the Dunford calculus, for all $s, t > 0$ we have

$$\varphi_\alpha(tA) \varphi_\beta(sA) = \lambda(s, t) \varphi_{\alpha+\beta}((s+t)A),$$

where $\lambda(s, t) = \frac{s^\alpha t^\beta}{(t+s)^{\alpha+\beta}}$. The following simple identity for functions $h \in L^2(\mathbb{R}_+, \frac{ds}{s})$ is the key to the proof:

$$\begin{aligned} \int_0^\infty \int_0^\infty |\lambda(s, t) h(t+s)|^2 \frac{ds}{s} \frac{dt}{t} &= \int_0^\infty \int_t^\infty \frac{(s-t)^{2\alpha-1} t^{2\beta}}{s^{2\alpha+2\beta}} |h(s)|^2 ds \frac{dt}{t} \\ &= \int_0^\infty \int_0^s \frac{(s-t)^{2\alpha-1} t^{2\beta}}{s^{2\alpha+2\beta}} |h(s)|^2 \frac{dt}{t} ds \\ &= \int_0^\infty \int_0^1 (1-t)^{2\alpha-1} t^{2\beta-1} |h(s)|^2 dt \frac{ds}{s} \\ &= C_{\alpha,\beta}^2 \int_0^\infty |h(s)|^2 \frac{ds}{s}. \end{aligned}$$

Stated differently, $h \mapsto C_{\alpha,\beta}^{-1} [(s, t) \mapsto \lambda(s, t) h(s+t)]$ is an isometric mapping from $L^2(\mathbb{R}_+, \frac{ds}{s})$ into $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \frac{ds}{s} \times \frac{dt}{t})$. By Theorem 9.6.1 it has an isometric γ -extension from $\gamma(\mathbb{R}_+, \frac{ds}{s}; X)$ into $\gamma(\mathbb{R}_+^2, \frac{ds}{s} \times \frac{dt}{t}; X)$. For all $x \in D(A) \cap R(A)$ the function $s \mapsto \varphi_{\alpha+\beta}(sA)x$ belongs to $\gamma(\mathbb{R}_+, \frac{ds}{s}; X)$ by Proposition 10.4.14 and therefore

$$\begin{aligned}
& \|s \mapsto \varphi_{\alpha+\beta}(sA)x\|_{\gamma(\mathbb{R}_+, \frac{ds}{s}; X)} \\
&= C_{\alpha, \beta} \|(s, t) \mapsto \lambda(s, t) \varphi_{\alpha+\beta}((s+t)A)x\|_{\gamma(\mathbb{R}_+^2, \frac{ds}{s} \times \frac{dt}{t}; X)} \\
&= C_{\alpha, \beta} \|(s, t) \mapsto \varphi_\alpha(tA) \varphi_\beta(sA)x\|_{\gamma(\mathbb{R}_+^2, \frac{ds}{s} \times \frac{dt}{t}; X)}.
\end{aligned}$$

It remains to estimate the last expression. Using Pisier's contraction property in the form (10.38), it follows that

$$\begin{aligned}
& \|(s, t) \mapsto \varphi_\alpha(tA) \varphi_\beta(sA)x\|_{\gamma(\mathbb{R}_+^2, \frac{ds}{s} \times \frac{dt}{t}; X)} \\
&\quad \approx_X \|t \mapsto [s \mapsto \varphi_\alpha(tA) \varphi_\beta(sA)x]\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; \gamma(\mathbb{R}_+, \frac{ds}{s}; X))} \\
&\quad \leq C_{0,A} \|s \mapsto \varphi_\beta(sA)x\|_{\gamma(\mathbb{R}_+, \frac{ds}{s}; X)} \leq C_{0,A} C_{1,A} \|x\|,
\end{aligned}$$

where the two inequalities follow from Lemma 10.4.29 and the assumption. The upper estimate of the lemma follows by combining the above estimates.

The lower estimate is proved in the same way, this time using the lower estimate of Lemma 10.4.29. $\square$

The next lemma, which is again based on Pisier's contraction property, will allow us to manipulate γ -norms.

Lemma 10.4.31. *Let X be a Banach space with Pisier's contraction property. Let ν be a σ -finite measure on $\mathbb{R}_+$ and assume that $t \mapsto [s \mapsto f(s, t)]$ belongs to $\gamma(\mathbb{R}_+, \nu; \gamma(\mathbb{R}_+, X))$. Then for all $g \in L^\infty(\mathbb{R}_+^2)$ the function $t \mapsto [s \mapsto g(s, t)f(s, t)]$ defines an element of $\gamma(\mathbb{R}_+, \nu; \gamma(\mathbb{R}_+, X))$ and*

$$\begin{aligned}
& \|s \mapsto [t \mapsto g(s, t)f(s, t)]\|_{\gamma(\mathbb{R}_+, \nu; \gamma(\mathbb{R}_+, X))} \\
&\quad \leq C_X \|g\|_\infty \|s \mapsto [t \mapsto f(s, t)]\|_{\gamma(\mathbb{R}_+, \nu; \gamma(\mathbb{R}_+, X))}.
\end{aligned}$$

Proof. Let λ denote the Lebesgue measure on $\mathbb{R}_+$. We begin by observing that for all $h \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \lambda \times \nu)$ the following estimate holds:

$$\int_0^\infty \int_0^\infty |g(s, t)h(s, t)|^2 ds d\nu(t) \leq \|g\|_\infty^2 \int_0^\infty \int_0^\infty |h(s, t)|^2 ds d\nu(t).$$

By Theorem 9.6.1 this estimate has the γ -extension

$$\begin{aligned}
& \|(s, t) \mapsto g(s, t)f(s, t)\|_{\gamma(\mathbb{R}_+^2, \lambda \times \nu; X)} \\
&\quad \leq \|g\|_\infty \|(s, t) \mapsto f(s, t)\|_{\gamma(\mathbb{R}_+^2, \lambda \times \nu; X)}.
\end{aligned}$$

Now the result follows by applying (10.38). $\square$

Proof of Theorem 10.4.28. The key point is to prove that A is R -sectorial with angle $\omega_R(A) < \frac{1}{2}\pi$. Once we know this, the remaining assertions follow from the earlier results in this chapter. Indeed, Theorem 10.4.9 then implies

that A has a bounded H^∞ -calculus of angle $\omega_{H^\infty}(A) \leq \omega_R(A)$, and then Corollary 10.4.10 gives that $\omega_{H^\infty}(A) = \omega_R(A)$.

To prove that A is R -sectorial, according to Proposition 10.3.3 it suffices to show that the sets

$$\{S(s) : s \geq 0\} \quad \text{and} \quad \{sAS(s) : s \geq 0\} \quad (10.39)$$

are R -bounded. Since X has Pisier's contraction property, it has finite cotype by Corollary 7.5.13, and therefore, γ -boundedness and R boundedness are equivalent by Theorem 8.1.3). Thus it suffices to check the γ -boundedness of the operator families in (10.39).

Repeated use of Lemma 10.4.30 gives that for all integers $k \geq 1$ and $x \in D(A) \cap R(A)$,

$$C^{-2k}\|x\| \leq \|s \mapsto \varphi_k(sA)x\|_{\gamma(\mathbb{R}_+, \frac{ds}{s}; X)} \leq C^{2k}\|x\|, \quad (10.40)$$

where C is as in Lemma 10.4.30. By Lemma 10.2.8 and the multiplicativity of the Dunford calculus applied to the function $(ze^{-z/k})^k = z^k e^{-z}$, we obtain $\varphi_k(sA) = s^k A^k S(s)$, where $(S(s))_{s \geq 0}$ denotes the analytic semigroup generated by $-A$. Therefore, (10.40) becomes

$$C^{-2k}\|x\| \leq \|s \mapsto s^{k-\frac{1}{2}} A^k S(s)x\|_{\gamma(\mathbb{R}_+; X)} \leq C^{2k}\|x\|. \quad (10.41)$$

Turning to the proof of the γ -boundedness of the set $\{S(s) : s \geq 0\}$, fix a finite sequence $(t_n)_{n=1}^N$ of non-negative real numbers and a sequence $(x_n)_{n=1}^N$ in $D(A) \cap R(A)$. Then applying the lower and upper estimate of (10.41) with $k = 1$ (in the first and last step) and Lemma 10.4.31 with $g(s, n) = \frac{s^{1/2}}{(s+t_n)^{1/2}}$ and $f(s, n) = s^{1/2}AS(s)x_n$, we obtain

$$\begin{aligned} & C^{-2} \left\| \sum_{n=1}^N \gamma_n S(t_n) x_n \right\|_{L^2(\Omega; X)} \\ & \leq \left\| s \mapsto \sum_{n=1}^N \gamma_n s^{1/2} AS(s+t_n) x_n \right\|_{L^2(\Omega; \gamma(\mathbb{R}_+; X))} \\ & = \left\| s \mapsto \left[n \mapsto \frac{s^{1/2}}{(s+t_n)^{1/2}} (s+t_n)^{1/2} AS(s+t_n) x_n \right] \right\|_{\gamma(\mathbb{R}_+; \gamma_N(X))} \\ & \leq C_X \left\| s \mapsto \left[n \mapsto s^{1/2} AS(s) x_n \right] \right\|_{\gamma(\mathbb{R}_+; \gamma_N(X))} \\ & = C_X \left\| s \mapsto s^{1/2} AS(s) \sum_{n=1}^N \gamma_n x_n \right\|_{L^2(\Omega; \gamma(\mathbb{R}_+; X))} \\ & \leq C^2 C_X \left\| \sum_{n=1}^N \gamma_n x_n \right\|_{L^2(\Omega; X)}. \end{aligned}$$

This shows that the set $\{S(s) : s \geq 0\}$ is γ -bounded, with constant $C^4 C_X$.

The proof of the γ -boundedness of $\{sAS(s) : s \geq 0\}$ proceeds in the same way, this time applying the lower and upper estimate of (10.41) with $k = 1$ and $k = 2$, respectively, and Lemma 10.4.31 with $g(s, n) = \frac{st_n}{(s+t_n)^2}$, where $f(s, n) = s^2 A^2 S(s)x_n$. With these choices,

$$\begin{aligned}
& C^{-2} \left\| \sum_{n=1}^N \gamma_n t_n A S(s_n) x_n \right\|_{L^2(\Omega; X)} \\
& \leq \left\| s \mapsto \sum_{n=1}^N \gamma_n t_n s^{1/2} A^2 S(s+t_n) x_n \right\|_{L^2(\Omega; \gamma(\mathbb{R}_+; X))} \\
& = \left\| s \mapsto \left[n \mapsto \frac{t_n s^{1/2}}{(s+t_n)^{3/2}} (s+t_n)^{3/2} A^2 S(s+t_n) x_n \right] \right\|_{\gamma(\mathbb{R}_+; \gamma_N(X))} \\
& \leq \frac{1}{3} C_X \left\| s \mapsto \left[n \mapsto s^{3/2} A^2 S(s) x_n \right] \right\|_{\gamma(\mathbb{R}_+; \gamma_N(X))} \\
& = \frac{1}{3} C_X \left\| s \mapsto s^{3/2} A^2 S(s) \sum_{n=1}^N \gamma_n x_n \right\|_{L^2(\Omega; \gamma(\mathbb{R}_+; X))} \\
& \leq \frac{1}{3} C^4 C_X \left\| \sum_{n=1}^N \gamma_n x_n \right\|_{L^2(\Omega; X)}.
\end{aligned}$$

This shows that the set $\{sAS(s) : s \geq 0\}$ is γ -bounded, with constant $\frac{1}{3} C^6 C_X$. $\square$

Remark 10.4.32.

(1) Instead of (10.40) one could use the following estimates in the above proof:

$$\begin{aligned}
C^{-1} \|x\| & \leq \|t \mapsto \varphi_{1/2}(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C \|x\| \\
& \|t \mapsto \varphi_{3/2}(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)} \leq C^3 \|x\|.
\end{aligned}$$

This gives the result with a better constant, but we haven't introduced fractional powers yet.

(2) Theorem 10.4.28 also holds if we assume the square function estimate holds for $\varphi_{1/n}(tA)$, where $n \geq 1$ is an integer instead of $\varphi_{1/2}(tA)$. To see this it suffices to observe that a repeated use of Lemma 10.4.30 still implies (10.40).

10.5 Necessity of UMD and Pisier's contraction property

This section is devoted to proving the necessity of the UMD property (resp. Pisier's contraction property) for the boundedness (resp. R -boundedness) of the imaginary powers of the negative Laplace operator $-\Delta$ on $L^p(\mathbb{R}^d; X)$. This

shows in particular the necessity of these assumption in Theorems 10.2.25 and 10.3.4(3). We begin with:

Theorem 10.5.1 (Guerre-Delabrière). *If the operators $(-\Delta)^{is}$ are uniformly bounded on $L_0^p(\mathbb{T}; X)$ for $s \in (0, 1]$, then X is a UMD space, and*

$$\beta_{p,X} \leq \liminf_{s \downarrow 0} \|(-\Delta)^{is}\|_{\mathcal{L}(L^p(\mathbb{T}; X))}.$$

Recall that $L_0^p(\mathbb{T}; X) := \{f \in L^p(\mathbb{T}; X) : \int f = 0\}$. Before going to the proof, we record:

Corollary 10.5.2 (Guerre-Delabrière). *If the operators $(-\Delta)^{is}$ are uniformly bounded on $L^p(\mathbb{R}^d; X)$ for $s \in (0, 1]$, then X is a UMD space, and*

$$\beta_{p,X} \leq \liminf_{s \downarrow 0} \|(-\Delta)^{is}\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))}.$$

In particular, this shows that the UMD assumption is necessary in Theorem 10.2.25, not only for the Laplacian to have functional calculus of angle 0 on $L^p(\mathbb{R}^d; X)$, but in fact for it to have a functional calculus of any angle at all. Indeed, the functions $f_s(z) = z^{is} = \exp(is \log z)$ are bounded and holomorphic on the largest possible sector $\Sigma_\pi = \mathbb{C} \setminus (-\infty, 0]$.

Proof. We observe that $(-\Delta)^{is}$ on $L^p(\mathbb{R}^d; X)$ is the Fourier multiplier T_m with multiplier $m(\xi) = (2\pi|\xi|)^{i2s}$; similarly, $(-\Delta)^{is}$ on $L_0^p(\mathbb{T}^d; X)$ is the periodic Fourier multiplier $\tilde{T}_{\tilde{m}}$ with multiplier $\tilde{m}(k) = (2\pi|k|)^{i2s}$, $k \in \mathbb{Z}^d \setminus \{0\}$.

Step 1: We apply a slight variant of the transference of Fourier multipliers given in Proposition 5.7.1. It says that if every $k \in \mathbb{Z}^d$ is a Lebesgue point of $m \in L^\infty(\mathbb{R}^d)$, then $(m(k))_{k \in \mathbb{Z}^d}$ is a Fourier multiplier on $L^p(\mathbb{T}^d; X)$ of at most the norm of the Fourier multiplier m on $L^p(\mathbb{R}^d; X)$. A slight obstacle is that 0 is not a Lebesgue point of our $m(\xi)$, no matter how we define $m(0)$. But, if we only consider the action of these operators on $L_0^p(\mathbb{T}^d; X)$, the 0th frequency never shows up, and one can check that the proof of Proposition 5.7.1 also applies, with trivial modifications, to the case that each $k \in \mathbb{Z}^d \setminus \{0\}$ is a Lebesgue point, giving in this case

$$\begin{aligned} \|\tilde{T}_{(m(k))_{k \in \mathbb{Z}^d \setminus \{0\}}}\|_{\mathcal{L}(L_0^p(\mathbb{T}^d; X))} &\leq \|T_m\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))}, \quad \text{i.e.} \\ \|(-\Delta)^{is}\|_{\mathcal{L}(L_0^p(\mathbb{T}^d; X))} &\leq \|(-\Delta)^{is}\|_{\mathcal{L}(L^p(\mathbb{R}^d; X))}. \end{aligned}$$

Step 2: Next we apply the operator $(-\Delta)^{is}$ on $L_0^p(\mathbb{T}^d; X)$ to functions of the first variable only, $f(t) = f_1(t_1)$, where $f_1 \in L_0^p(\mathbb{T}; X)$. These have non-zero Fourier coefficients only for $k = (k_1, 0, \dots, 0)$, in which case $(2\pi|k|)^{i2s} = (2\pi|k_1|)^{i2s}$, and we see that $((-\Delta)^{is}f)(t) = ((-\Delta_1)^{is}f)(t_1)$, where Δ_1 is the one-dimensional Laplacian on $\mathbb{T}$. Thus

$$\|(-\Delta)^{is}\|_{\mathcal{L}(L_0^p(\mathbb{T}; X))} \leq \|(-\Delta)^{is}\|_{\mathcal{L}(L_0^p(\mathbb{T}^d; X))}.$$

Combining the above displayed estimates with Theorem 10.5.1 gives the statement of the corollary. $\square$

The proof of Theorem 10.5.1 is based on the fact that the values of the multiplier $|\xi|^{is}$ provide good approximations of any prescribed numbers $|\varepsilon_k| = 1$ on long intervals $I_k \ni \xi$:

Lemma 10.5.3. *Given a sequence $s_j \downarrow 0$, numbers $\delta > 0$ and $L \geq 2$, and complex signs $|\epsilon_k| = 1$ for $1 \leq k \leq K$, we can find integers $N_0 = 0 < N_1 < N_2 < \dots < N_K$ and $s = s_j$ in such a way that*

$$||\xi|^{is} - \epsilon_k| \leq \delta \quad \forall \xi \in I_k := [N_k - LN_{k-1}, LN_k], \quad k = 1, \dots, K.$$

Proof. For $\xi \in I_k$, we have

$$\log |\xi| - \log N_k \in \left[\log \left(1 - \frac{LN_{k-1}}{N_k} \right), \log L \right] \subseteq \left[\log \frac{1}{2}, \log L \right] \subseteq \left[-\frac{\delta}{2s}, \frac{\delta}{2s} \right],$$

provided that $N_k > 2LN_{k-1}$ and $s = s_j$ is small enough, depending only on L (indeed, we need $s \leq \delta/(2 \log L)$). We fix such an s , and proceed to specify the choice of N_k .

Let $N_0 := 0$. Assuming that N_{k-1} is already chosen, the choice of N_k is as follows. Write $\epsilon_k = e^{it_k}$. Since $(\log n)_{n \geq 1}$ increases to ∞ with decreasing steps $\log(n+1) - \log n \leq \frac{1}{n} \rightarrow 0$, we find that $s \log n$ comes infinitely often arbitrarily close to $t_k + 2\pi\mathbb{Z}$. Thus we can pick $N_k > 2LN_{k-1}$ such that

$$|s \log N_k - (t_k + 2\pi m_k)| \leq \delta/2$$

for some $m_k \in \mathbb{Z}$. But then, for $\xi \in I_k$, we have

$$\begin{aligned} ||\xi|^{is} - \epsilon_k| &= |e^{is \log |\xi|} - e^{i(t_k + 2\pi m_k)}| \\ &\leq s |\log |\xi| - \log N_k| + |s \log N_k - (t_k + 2\pi m_k)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

□

Proof of Theorem 10.5.1. We begin by exactly the same reduction as in the proof of Theorem 5.2.10 (the necessity of UMD for the Hilbert transform's boundedness); but we repeat this short step for the reader's convenience: By Theorem 4.2.5 it suffices to estimate the dyadic UMD constant. In order to most conveniently connect this with Fourier analysis, we choose a model of the Rademacher system $(r_k)_{k=1}^n$, where the probability space is $\mathbb{T}^n = \mathbb{T}_1 \times \dots \times \mathbb{T}_n$ (each $\mathbb{T}_k$ is simply an indexed copy of $\mathbb{T}$), and $r_k = r_k(t_k)$ is a function of the k th coordinate only. Then it is (more than) sufficient to prove that

$$\left\| \sum_{k=1}^n \epsilon_k f_k \right\|_{L^p(\mathbb{T}^n; X)} \leq \liminf_{s \downarrow 0} \|(-\Delta)^{is}\|_{\mathcal{L}(L^p(\mathbb{T}; X))} \left\| \sum_{k=1}^n f_k \right\|_{L^p(\mathbb{T}^n; X)}$$

for all $f_k \in L_0^p(\mathbb{T}_k; L^p(\mathbb{T}^{k-1}; X))$, since in particular each $\phi_k(r_1, \dots, r_{k-1})r_k$ has this form, and the latter are precisely the martingale differences of

Paley–Walsh martingales (see Proposition 3.1.10). We use the convention that $L^p(\mathbb{T}^0; X) := X$. By the density of trigonometric polynomials in L^p , we may further assume that each f_k is a trigonometric polynomial on $\mathbb{T}_k$, with values in the space of trigonometric polynomials on $\mathbb{T}^{k-1}$.

Fix some $(t_1, \dots, t_n) \in \mathbb{T}^n$, and denote $\mathbf{t}_k := (t_1, \dots, t_k) \in \mathbb{T}^k$. We also consider integers $0 = N_0 < N_1 < \dots < N_n$, and denote $\mathbf{N}_k := (N_1, \dots, N_k) \in \mathbb{Z}^k$. Since f_k lacks the zeroth frequency as a function of t_k , the function

$$t \in \mathbb{T} \mapsto F_k(t) := f_k(\mathbf{N}_k t + \mathbf{t}_k) = f_k(\mathbf{N}_{k-1} t + \mathbf{t}_{k-1}, N_k t + t_k)$$

has non-zero Fourier coefficients only for frequencies

$$|\xi| \in I_k := [N_k - LN_{k-1}, LN_k],$$

where L is a number depending only on the frequencies of the trigonometric polynomials $f_1, \dots, f_n$. (By the assumption that $(N_k)_{k=1}^n$ is increasing, we may absorb the dependence on $\mathbf{N}_k$ into a dependence on N_k only.)

Let us denote $\|f\|_A = \|f\|_{A(\mathbb{T}^k; X)} := \sum_{\xi \in \mathbb{Z}^k} \|\hat{f}(\xi)\|_X$, and observe that $(-\Delta)^{is}$ is the Fourier multiplier with symbol $(2\pi|\xi|)^{i2s} = (2\pi)^{i2s}|\xi|^{i2s}$. If we now choose the integers N_k and the number $s/2 > 0$ (in place of s) as in Lemma 10.5.3, this lemma guarantees that

$$\|(-\Delta)^{is/2} F_k - (2\pi)^{is} \epsilon_k F_k\|_A \leq \delta \|F_k\|_A \leq \delta \|f_k\|_A;$$

indeed, this holds for the trigonometric monomial e_ξ with $|\xi| \in I_k$, and hence for the A -norm of F_k by summing over all $|\xi| \in I_k$.

Thus

$$\left\| \sum_{k=1}^n \epsilon_k F_k \right\|_p = \left\| \sum_{k=1}^n (2\pi)^{is/2} \epsilon_k F_k \right\|_p \leq \left\| \sum_{k=1}^n (-\Delta)^{is/2} F_k \right\|_p + \delta \sum_{k=1}^n \|f_k\|_A$$

and, noting that all F_k have integral zero,

$$\left\| \sum_{k=1}^n (-\Delta)^{is/2} F_k \right\|_p \leq \|(-\Delta)^{is/2}\|_{\mathcal{L}(L_0^p(\mathbb{T}; X))} \left\| \sum_{k=1}^n F_k \right\|_p.$$

Taking the $L^p(\mathbb{T}^n)$ -norms with respect to $\mathbf{t}_n \in \mathbb{T}^n$, we note that

$$\begin{aligned} \left\| \sum_{k=1}^n \epsilon_k F_k \right\|_{L^p(\mathbb{T}^n; L^p(\mathbb{T}; X))} &= \left\| \sum_{k=1}^n \epsilon_k f_k(t \mathbf{N}_k + \mathbf{t}_k) \right\|_{L^p(\mathbb{T}, dt; L^p(\mathbb{T}^n, d\mathbf{t}_n; X))} \\ &= \left\| \sum_{k=1}^n \epsilon_k f_k(\mathbf{t}_k) \right\|_{L^p(\mathbb{T}, dt; L^p(\mathbb{T}^n, d\mathbf{t}_n; X))} = \left\| \sum_{k=1}^n \epsilon_k f_k \right\|_{L^p(\mathbb{T}^n; X)} \end{aligned}$$

by Fubini's theorem and translation invariance of the $L^p(\mathbb{T}^n; X)$ -norm. Substituting back, we have checked that

$$\left\| \sum_{k=1}^n \epsilon_k f_k \right\|_p \leq \|(-\Delta)^{is/2}\|_{\mathcal{L}(L_0^p(\mathbb{T}; X))} \left\| \sum_{k=1}^n f_k \right\|_p + \delta \sum_{k=1}^n \|f_k\|_A.$$

As the only dependence on δ here is the explicit factor in the last term, we may pass to the limit $\delta \rightarrow 0$ to eliminate this term. Finally, we observe that Lemma 10.5.3 allowed us to choose s (and then also $s/2$) as any number from a given sequence $s_j \rightarrow 0$ and hence we may finally pass to the limes inferior along such a sequence to deduce that

$$\left\| \sum_{k=1}^n \epsilon_k f_k \right\|_p \leq \liminf_{s \rightarrow 0} \|(-\Delta)^{is}\|_{\mathcal{L}(L_0^p(\mathbb{T}; X))} \left\| \sum_{k=1}^n f_k \right\|_p,$$

as claimed. $\square$

There is a variant involving R -boundedness and Pisier's contraction property:

Proposition 10.5.4. *If the operators $(-\Delta)^{is}$ are R -bounded on $L_0^p(\mathbb{T}; X)$ for $s \in (0, 1]$, then X has Pisier's contraction property, and*

$$\alpha_{p,X} \leq \lim_{t \downarrow 0} \mathcal{R}_p((-\Delta)^{is} : s \in (0, t)).$$

Corollary 10.5.5. *If the operators $(-\Delta)^{is}$ are R -bounded on $L^p(\mathbb{R}^d; X)$ for $s \in (0, 1]$, then X has Pisier's contraction property, and*

$$\alpha_{p,X} \leq \lim_{t \downarrow 0} \mathcal{R}_p((-\Delta)^{is} : s \in (0, t)).$$

This shows in particular that the assumption that X has Pisier's contraction property is necessary in Theorem 10.3.4(3).

Proof. The deduction of this corollary from Proposition 10.5.4 via transference is very similar to the deduction of Corollary 10.5.2 from Theorem 10.5.1, e.g. after reformulating the R -boundedness as the boundedness of suitable operator-valued multipliers on L^p -spaces of $\varepsilon_N^p(X)$ -valued functions as in the proof of Proposition 8.3.24. The details are left to the reader. $\square$

For the proof of Proposition 10.5.4, we borrow a piece of ergodic theory:

Lemma 10.5.6. *Let $s_1, \dots, s_n \in \mathbb{R}$ and 1 be linearly independent over $\mathbb{Q}$, and let $\mathbf{s} := (s_1, \dots, s_n)$. Then for every $x \in \mathbb{T}^n$, the orbit $\{x + m\mathbf{s} \bmod 1 : m \in \mathbb{N}\}$ is dense in $\mathbb{T}^n$.*

Proof. Step 1: A zero/one law. We first check that if a measurable $E \subseteq \mathbb{T}^n$ is invariant under $x \mapsto x + \mathbf{s} \bmod 1$, then $|E| \in \{0, 1\}$. Indeed, writing the Fourier series of $\mathbf{1}_E(t) = \mathbf{1}_{E+\mathbf{s}}(t) = \mathbf{1}_E(t - \mathbf{s})$ in two ways, we find (with convergence in $L^2(\mathbb{T}^n)$) that

$$\sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{1}_E}(k) e_k(t) = \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{1}_E}(k) e_k(t - \mathbf{s}) = \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{1}_E}(k) e^{-i2\pi k \cdot \mathbf{s}} e_k(t),$$

and by the uniqueness of the Fourier series that $\widehat{\mathbf{1}_E}(k)(e^{-i2\pi k \cdot \mathbf{s}} - 1) = 0$ for all $k \in \mathbb{Z}^n$. The second factor vanishes if and only if $k \cdot \mathbf{s}$ is an integer, but the assumed $\mathbb{Q}$ -linear independence of $s_1, \dots, s_n$ and 1 implies that this can only happen when $k = 0$. Hence we conclude that $\widehat{\mathbf{1}_E}(k) = 0$ for all $k \in \mathbb{Z}^n \setminus \{0\}$, and thus $\mathbf{1}_E(t) = \widehat{\mathbf{1}_E}(0)$ is a constant almost everywhere. For an indicator function, this leaves the two options $|E| \in \{0, 1\}$.

Step 2: Density of $x + \mathbb{Z}\mathbf{s}$. Assume for contradiction that $\{x + m\mathbf{s} : m \in \mathbb{Z}\}$, for some $x \in \mathbb{T}^n$, avoids a point $y \in \mathbb{T}^n$ by a positive distance $3\delta > 0$. But then $U := \{x' + m\mathbf{s} : m \in \mathbb{Z}^n, x' \in B(x, \delta)\}$ avoids $V := \{y' + m\mathbf{s} : m \in \mathbb{Z}^n, y' \in B(y, \delta)\}$ by a positive distance $\delta > 0$. Clearly both U and V are invariant under $x \mapsto x + \mathbf{s}$ and measurable with positive measure (as countable unions of balls in $\mathbb{T}^n$). On the other hand, being disjoint, neither can have full measure in $\mathbb{T}^n$. But this contradicts the zero/one law just established, thereby showing that $\{x + m\mathbf{s} : m \in \mathbb{Z}\}$ must come arbitrarily close to every $y \in \mathbb{T}^n$.

Step 3: Density of $x + \mathbb{N}\mathbf{s}$ for some x . Fix a point $x_0 \in \mathbb{T}^n$ and let $x_k := x_0 - k\mathbf{s}$ for $k = 0, 1, 2, \dots$. Then there exists a subsequence x_{k_j} with a limit $x_\infty \in \mathbb{T}^n$. We show that $x_\infty + \mathbb{N}\mathbf{s}$ is dense. Let $y \in \mathbb{T}^n$ and $\delta > 0$ be arbitrary. Since $x_0 + \mathbb{Z}\mathbf{s}$ is dense, we can find some $m \in \mathbb{Z}$ such that $x_0 + m\mathbf{s} \in B(y, \delta/2)$. Next, we choose $k = k_j \geq |m|$ so large that $|x_k - x_\infty| \leq \delta/2$. But then $x_\infty + (m+k)\mathbf{s} \in x_\infty + \mathbb{N}\mathbf{s}$ is within $\delta/2$ from $x_k + (m+k)\mathbf{s} = x_0 + m\mathbf{s} \in B(y, \delta/2)$, and hence $x_\infty + \mathbb{N}\mathbf{s}$ comes within distance δ from y . Since δ and y were arbitrary, this proves the density of $x_\infty + \mathbb{N}\mathbf{s}$.

Step 4: Conclusion. If $x \in \mathbb{T}^n$ is the point constructed in Step 3, and $x' \in \mathbb{T}^n$ is arbitrary, then

$$\overline{x' + \mathbb{N}\mathbf{s}} = \overline{(x' - x) + (x + \mathbb{N}\mathbf{s})} = (x' - x) + \overline{x + \mathbb{N}\mathbf{s}} = (x' - x) + \mathbb{T}^n = \mathbb{T}^n,$$

and hence $x' + \mathbb{N}\mathbf{s}$ is also dense. $\square$

Proof of Proposition 10.5.4. Applying the R -boundedness of $(-\Delta)^{is}$ to N functions of the form $f_j = \sum_{k=1}^N \varepsilon'_k e_{n_k} x_{jk}$, we deduce that

$$\begin{aligned} & \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k e_{n_k} (2\pi |n_k|)^{is_j} x_{jk} \right\|_{L^p(\Omega \times \mathbb{T}; X)} \\ & \leq \mathcal{R}_p((-\Delta)^{is_j/2} : 1 \leq j \leq N) \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k e_{n_k} x_{jk} \right\|_{L^p(\Omega \times \mathbb{T}; X)}. \end{aligned}$$

Since $(\varepsilon'_k e_{k_j}(t))_{k=1}^N$ and $(\varepsilon'_k)_{k=1}^N$ are equidistributed for each $t \in \mathbb{T}$, we may further eliminate the factors e_{k_j} and the integration over $\mathbb{T}$ in the L^p -norm. Similarly, since $((2\pi)^{is_j} \varepsilon_j)_{j=1}^N$ and $(\varepsilon_j)_{j=1}^N$ are equidistributed, we may also eliminate the factors $(2\pi)^{is_j}$. The resulting estimate is essentially Pisier's contraction property, up to checking that the factors $|n_k|^{is_j}$ can be chosen to approximate an arbitrary double array of numbers α_{jk} of modulus one.

To this end, we first pick $0 < s_1, \dots, s_n < t$ such that $s_1, \dots, s_n$ and 1 are linearly independent over $\mathbb{Q}$, and Lemma 10.5.6 applies. We write $\alpha_{jk} = e^{it_{jk}}$

for some $t_{jk} \in \mathbb{T}$. Let $\mathbf{s} := (s_1, \dots, s_n)$, and $\mathbf{t}_k := (t_{1k}, \dots, t_{nk}) \in \mathbb{T}^n$. We want to choose $N_1, \dots, N_n \in \mathbb{Z}_+$ such that

$$|\mathbf{s} \log N_k - \mathbf{t}_k| \leq \delta, \quad k = 1, \dots, n.$$

Let first N'_k be so large that $|\mathbf{s}|/N'_k \leq \delta/2$. Then apply Lemma 10.5.6 to find $n_k \in \mathbb{N}$ such that

$$|\mathbf{s}(\log N'_k + n'_k) - \mathbf{t}_k| \leq \delta/2.$$

Since $\log(n+1) - \log n \leq 1/n \leq 1/N'_k$ for $n \geq N'_k$, we can find an integer $N_k \geq N'_k$ such that $|\log N_k - (\log N'_k + n_k)| \leq 1/N'_k$, and hence

$$\begin{aligned} |\mathbf{s} \log N_k - \mathbf{t}_k| &\leq |\mathbf{s}| |\log N_k - (\log N'_k + n_k)| + |\mathbf{s}(\log N'_k + n'_k) - \mathbf{t}_k| \\ &\leq |\mathbf{s}|/N'_k + \delta/2 \leq \delta/2 + \delta/2 = \delta, \end{aligned}$$

as required.

Now we can estimate

$$\begin{aligned} \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k \alpha_{jk} x_{jk} \right\|_{L^p(\Omega; X)} &= \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k e^{it_{jk}} x_{jk} \right\|_{L^p(\Omega; X)} \\ &\leq \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k |N_k|^{is_j} x_{jk} \right\|_{L^p(\Omega; X)} + \sum_{j,k=1}^N |s_j \log |N_k| - t_{jk}| \|x_{jk}\|_X \\ &\leq \mathcal{R}_p((-\Delta)^{is/2} : 0 < s < t) \left\| \sum_{j,k=1}^N \varepsilon_j \varepsilon'_k x_{jk} \right\|_{L^p(\Omega; X)} + \delta \sum_{j,k=1}^N \|x_{jk}\|_X. \end{aligned}$$

Since $\delta > 0$ and the array $(x_{jk})_{j,k=1}^N$ is arbitrary, we conclude that

$$\alpha_{p,X} \leq \mathcal{R}_p((-\Delta)^{is/2} : 0 < s < t).$$

Finally, $t > 0$ was also arbitrary, so we can pass to the limit $t \downarrow 0$. $\square$

10.6 The bisectorial H^∞ -calculus

In this section we discuss a variant of the H^∞ -calculus for a class of operators, whose spectrum is spread over two opposite sectors of the complex plane. Throughout this chapter we shall work over the complex scalar field.

10.6.a Bisectorial operators

For $0 < \omega < \frac{1}{2}\pi$ we define $\Sigma_\omega^+ := \Sigma_\omega$ and $\Sigma_\omega^- := -\Sigma_\omega$, and

$$\Sigma_\omega^{\text{bi}} := \Sigma_\omega^+ \cup \Sigma_\omega^-.$$

Here, as always, Σ_ω is the open sector about the positive real axis of opening angle ω . The set $\Sigma_\omega^{\text{bi}}$ is called the *bisector* of angle ω .

Definition 10.6.1. A linear operator A on a Banach space X is said to be bisectorial if there exists an $\omega \in (0, \frac{1}{2}\pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega^{\text{bi}}}$ and

$$M_{\omega,A}^{\text{bi}} := \sup_{z \in \overline{\Sigma_\omega^{\text{bi}}}} \|zR(z, A)\| < \infty.$$

In this situation we say that A is ω -bisectorial. The infimum of all $\omega \in (0, \frac{1}{2}\pi)$ such that A is ω -bisectorial is called the angle of bisectoriality of A and is denoted by $\omega^{\text{bi}}(A)$.

Replacing the uniform boundedness condition on the operators $zR(z, A)$ by an R -boundedness condition we obtain the notion of an R -bisectorial operator; the corresponding R -bounds will be denoted by $M_{\omega,A}^{\text{bi},R}$.

Fig. 10.2: The spectrum of a bisectorial operator

By Proposition 10.1.6 a linear operator A is bisectorial if and only if $i\mathbb{R} \setminus \{0\} \subseteq \varrho(A)$ and

$$\sup_{t \in \mathbb{R}_+ \setminus \{0\}} \|tR(it, A)\| < \infty.$$

Let us record some immediate consequences of the definition.

- An operator A is bisectorial if and only both operators $\pm iA$ are sectorial.
- A sectorial operator of angle less than $\frac{1}{2}\pi$ is bisectorial of the same angle.
- If A is bisectorial of angle less than $\frac{1}{2}\pi$ and $\sigma(A) \subseteq \overline{\mathbb{C}_+}$, then A is sectorial of the same angle under moderate growth conditions on the resolvent near 0 and infinity. Indeed, by assumption we have uniform bounds for $zR(z, A)$ on appropriate sectors around $\pm i\mathbb{R}$. The remaining uniform bound for $zR(z, A)$ in the remaining sector containing the negative real line follows from this via the sectorial version of the three lines lemma.

In view of these facts it is to be expected that much of the theory of sectorial operators carries over to, or has an analogue for, bisectorial operators. This is indeed the case. In particular this is true for Propositions 10.1.7, 10.1.8, and 10.1.9. Their bisectorial analogues will be used without further comment.

The following result establishes a simple connection between the bisectoriality of an operator A and the sectoriality of its square A^2 .

Proposition 10.6.2. *Let $0 < \sigma < \frac{1}{2}\pi$. If A is σ -bisectorial (resp. σ - R -bisectorial), then:*

(1) A^2 is 2σ -sectorial (resp. 2σ - R -sectorial) and

$$M_{2\sigma, A^2}^{\text{bi}} \leq (M_{\sigma, A})^2 \quad (\text{resp. } M_{2\sigma, A^2}^{\text{bi}, R} \leq (M_{\sigma, A}^R)^2);$$

(2) $\overline{R(A^2)} = \overline{R(A)}$ and $N(A^2) = N(A)$.

Proof. If both $\pm z \in \varrho(A)$, then $z^2 \in \varrho(A^2)$ and

$$(z^2 - A^2)^{-1} = -(z - A)^{-1}(-z - A)^{-1}.$$

The (R)-resolvent bound for A^2 also follows from this identity. This proves (1).

To prove the first assertion in (2) let $x \in \overline{R(A)}$. It follows from (10.5) (noting that iA is sectorial) that $x = \lim_{s \rightarrow 0} A(is + A)^{-1}x = \lim_{s \rightarrow 0} \lim_{t \rightarrow 0} A^2(it + A)^{-1}(is + A)^{-1}x$, so $\overline{R(A)} \subseteq \overline{R(A^2)}$. The reverse inclusion is trivial.

If $x \in N(A^2)$, then by (10.4) (again noting that iA is sectorial), $Ax = \lim_{t \rightarrow 0} (it + A)^{-1}A^2x = 0$, so $N(A^2) \subseteq N(A)$. The reverse inclusion is trivial. $\square$

Examples

The first example is of abstract nature and plays an important role in the transference results of Section 10.7.b.

Example 10.6.3 (*Generators of bounded C_0 -groups*). If iA generates a bounded C_0 -group (see Appendix G), then A is bisectorial of angle 0. This is immediate from the easy part of the Hille–Yosida theorem, which provides (after a rotation about $\frac{1}{2}\pi$) the resolvent estimate $\|R(\lambda, iA)\| \leq M_\sigma/|\Re \lambda|$, valid for all $0 < \sigma < \frac{1}{2}\pi$ and $\lambda \in \pm\Sigma_\sigma^{\text{bi}}$.

Example 10.6.4 (*The operators $\pm id/dx$*). As a special case of the preceding example, the operators $\pm id/dx$ are bisectorial on $L^p(\mathbb{R}; X)$ of angle 0, for all $1 < p < \infty$ and all Banach spaces X .

Example 10.6.5 (*The Hodge–Dirac operator*). On $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d; \mathbb{C}^d)$ we consider the operator

$$D := \begin{pmatrix} 0 & \nabla^* \\ \nabla & 0 \end{pmatrix}$$

with its natural domain $D(D) = D(\nabla) \oplus D(\nabla^*)$, where $D(\nabla)$ is the Sobolev space $W^{1,2}(\mathbb{R}^d)$ and $D(\nabla^*)$ is the domain of the Hilbert space adjoint $\nabla^* = -\operatorname{div} \nabla$. Standard arguments show that D is self-adjoint and therefore, by Stone's theorem (Theorem G.6.2), iD generates a unitary C_0 -group. Example 10.6.3 then gives the bisectoriality of D .

The interest of this operator derives from the fact that it “squares to $-\Delta$ ” in the sense that its square is of the form

$$D^2 = \begin{pmatrix} -\Delta & 0 \\ 0 & * \end{pmatrix}.$$

10.6.b Basic theory of the bisectorial calculus

The construction of the Dunford calculus of a bisectorial operator A is analogous to the sectorial case, the only difference being that we replace the Hardy spaces $H^p(\Sigma)$ by their bisectorial counterparts $H^p(\Sigma^{\text{bi}})$, the Banach spaces of all holomorphic functions $f : \Sigma_\sigma^{\pm \text{bi}} \rightarrow \mathbb{C}$ for which the norm

$$\|f\|_{H^p(\Sigma_\sigma^{\text{bi}})} := \sup_{|\nu| < \sigma} \|t \mapsto f(e^{i\nu}t)\|_{L^p(\mathbb{R}_+, \frac{dt}{t})}$$

is finite. We define the Dunford integrals by integrating over the boundary of Σ^{bi} . More precisely, for $f \in H^1(\Sigma_\sigma^{\text{bi}})$ with $\omega^{\text{bi}}(A) < \sigma < \frac{1}{2}\pi$ we set

$$f(A) := \int_{\partial \Sigma_\nu^{\text{bi}}} f(z) R(z, A) dz,$$

where $\omega^{\text{bi}}(A) < \nu < \sigma$ is chosen arbitrarily. The path of integration is traversed so that the interior of the two sectors $\Sigma_\nu^\pm$ is always at the left hand. We say that A has a bounded $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus if there exists a constant $M \geq 0$ such that

$$\|f(A)\| \leq M \|f\|_\infty \quad f \in H^1(\Sigma_\sigma^{\text{bi}}) \cap H^\infty(\Sigma_\sigma^{\text{bi}}).$$

Proposition 10.2.11 carries over to the bisectorial case *mutatis mutandis*, with only one small difference: we now set

$$\tilde{\zeta}(z) := \frac{z}{z^2 + 1} = \frac{z}{(z+i)(z-i)},$$

then check that $\tilde{\zeta}(A) = A(A+i)^{-1}(A-i)^{-1}$ and define, for $f \in H^\infty(\Sigma_\sigma^{\text{bi}})$ and $x \in D(A) \cap R(A)$, say $x = \tilde{\zeta}(A)y$,

$$\Psi_A(f)x := (f\tilde{\zeta})(A)y.$$

Proposition 10.6.6. *Let A be a bisectorial operator on X and let $\omega^{\text{bi}}(A) < \sigma < \frac{1}{2}\pi$. Let $C > 0$. The following assertions are equivalent:*

- (1) A has a bounded $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus;

(2) there is a constant $C \geq 0$ such that for all $f \in H^1(\Sigma_\sigma^{\text{bi}}) \cap H^\infty(\Sigma_\sigma^{\text{bi}})$,

$$\|f(A)x\| \leq C\|f\|_\infty\|x\|, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A);$$

(3) there is a constant $C \geq 0$ such that for all $f \in H^\infty(\Sigma_\sigma^{\text{bi}})$,

$$\|\Psi_A(f)x\| \leq C\|f\|_\infty\|x\|, \quad x \in \mathcal{D}(A) \cap \mathcal{R}(A).$$

Moreover, analogous relations between the best constants hold. The best constant in (3) is denoted by $M_{\sigma,A}^{\text{bi},\infty}$.

As in the sectorial case the operators $\Psi_A(f)$ map $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ into itself and the resulting mapping

$$\Psi_A : H^\infty(\Sigma_\sigma^{\text{bi}}) \mapsto \mathcal{L}(\overline{\mathcal{D}(A) \cap \mathcal{R}(A)})$$

is called the $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus of A . It shares the properties collected in Theorem 10.2.13: the calculus is additive, for $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ we have

$$\mathbf{1}(A)x = x \quad \text{and} \quad r_\mu(A)x = R(\mu, A)x$$

for $r_\mu(z) = (\mu - z)^{-1}$ and $\mu \in \mathbb{C} \setminus \overline{\Sigma_\sigma^{\text{bi}}}$, and it enjoys the bisectorial analogue of the convergence property (10.16). These properties will be used in the sequel without further comment.

The next theorem establishes an important connection between the H^∞ -calculus of a bisectorial operator A and the sectorial operator A^2 . The proof makes use of the fact, which we leave to the reader to verify, that the square function characterisations of the H^∞ -calculus of Section 10.4 carry over to the bisectorial case *mutatis mutandis*. In the present chapter we will only need the elementary part of the theorem, which is the implication (1) $\Rightarrow$ (2). The power of the theorem, however, resides in the implication (2) $\Rightarrow$ (1), which often allows one to study Riesz transforms by means of functional calculus methods.

Theorem 10.6.7. *Let A be an R -bisectorial operator on a Banach space. Then A^2 is R -sectorial, and for all $\omega^{\text{bi}}(A) < \sigma < \frac{1}{2}\pi$ the following assertions hold:*

- (1) *if A admits a bounded $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus, then A^2 admits a bounded $H^\infty(\Sigma_{2\sigma})$ -calculus and $M_{2\sigma,A^2}^{\text{bi},\infty} \leq M_{\sigma,A}^\infty$;*
- (2) *if A^2 admits a bounded $H^\infty(\Sigma_{2\sigma})$ -calculus, then A admits a bounded $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus for all $\sigma < \vartheta < \pi$.*

Proof. The R -sectoriality of A^2 with angle $2\omega_R(A)$ has already been observed in Proposition 10.6.2.

(1): To see that A^2 admits a bounded $H^\infty(\Sigma_{2\sigma})$ -calculus, let $f \in H^1(\Sigma_{2\sigma}) \cap H^\infty(\Sigma_{2\sigma})$ be given and compute

$$\begin{aligned}
f(A^2) &= \frac{1}{2\pi i} \int_{\partial \Sigma_{2\nu}^+} f(z) R(z, A^2) dz \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}^+} f(w^2) R(w^2, A^2) 2w dw \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}^+} f(w^2) [R(w, A) - R(-w, A)] dw \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}^{\text{bi}}} f(w^2) R(w, A) dw = \tilde{f}(A),
\end{aligned} \tag{10.42}$$

where $\tilde{f}(z) := f(z^2)$. Thus

$$\|f(A^2)\|_{\mathcal{L}(X)} = \|\tilde{f}(A)\|_{\mathcal{L}(X)} \leq M_{\nu, A}^{\infty} \|\tilde{f}\|_{H^{\infty}(\Sigma_{\sigma}^{\text{bi}})} = M_{\nu, A}^{\infty} \|f\|_{H^{\infty}(\Sigma_{2\sigma}^+)}.$$

It follows that A^2 has a bounded $H^{\infty}(\Sigma_{2\sigma})$ -calculus. By Letting $\nu \uparrow \sigma$, this also gives the bound on the constants.

(2): Fix an arbitrary non-zero function $g \in H^1(\Sigma_{2\vartheta}) \cap H^{\infty}(\Sigma_{2\vartheta})$ with $\sigma < \vartheta < \pi$, and define $\tilde{g} \in H^1(\Sigma_{\vartheta}^{\text{bi}}) \cap H^{\infty}(\Sigma_{\vartheta}^{\text{bi}})$ by $\tilde{g}(z) := g(z^2)$. Then $\tilde{g}$ is non-zero on both $\Sigma_{\vartheta}^{\pm}$.

Since A^2 has a bounded $H^{\infty}(\Sigma_{2\sigma})$ -functional calculus, by Theorem 10.4.4 it satisfies a discrete square function estimate

$$\sup_{t>0} \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n g(2^n t A^2) x \right\|^2 \right)^{1/2} \approx \|x\|$$

for $x \in \overline{\mathcal{R}(A^2) \cap \mathcal{D}(A^2)} = \overline{\mathcal{R}(A) \cap \mathcal{D}(A)}$. By (10.42) we have $g(tA^2) = \tilde{g}(\sqrt{t}A)$, and therefore by the substitution $\sqrt{t} = s$ the above estimate is equivalent to

$$\sup_{s>0} \sup_{N \geq 1} \left(\mathbb{E} \left\| \sum_{|n| \leq N} \varepsilon_n \tilde{g}(\sqrt{2}^n s A) x \right\|^2 \right)^{1/2} \approx \|x\|$$

for $x \in \overline{\mathcal{R}(A^2) \cap \mathcal{D}(A^2)} = \overline{\mathcal{R}(A) \cap \mathcal{D}(A)}$. By the bisectorial version of Theorem 10.4.9, this estimate implies that A has a bounded $H^{\infty}(\Sigma_{\sigma}^{\text{bi}})$ -calculus. $\square$

10.7 Functional calculus for (semi)group generators

Having studied the properties of the H^{∞} -calculus in fair detail from an abstract point of view, in this final section of this chapter we turn to some concrete situations where the boundedness of the H^{∞} -calculus can actually be verified. As we shall see, this often involves importing deep results from other areas of analysis. This should not come as a surprise, since after all proving the boundedness of the H^{∞} -calculus is tantamount to controlling singular integrals. We have already seen one explicit instance when we deduced the boundedness of the H^{∞} -calculus for the Laplace operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, from the Mihlin multiplier theorem. From this point of view the highlights of this section will be the following results:

- if iA is the generator of a uniformly bounded C_0 -group on a UMD Banach space X , then A has a bounded bisectorial H^∞ -calculus of angle 0 (Theorem 10.7.10).
- if $-A$ is the generator of a positive C_0 -semigroup of contractions on an L^p -space with $1 < p < \infty$, then A has a bounded H^∞ -calculus of angle $\leq \frac{1}{2}\pi$ (Theorem 10.7.12).
- if $-A$ is the generator of a positive C_0 -semigroup of contractions on an L^p -space with $1 < p < \infty$, and if this semigroup is bounded and holomorphic on a sector of angle $\eta \in (0, \frac{1}{2}\pi)$, then A has a bounded H^∞ -calculus of angle $\leq \frac{1}{2}\pi - \frac{1}{2}\eta(p \wedge p')$ (Theorem 10.7.13).

The proofs of these theorems depend on transference techniques due to Coifman and Weiss in combination with, respectively, (a special case of) the Akcoglu–Sucheston dilation theorem for positive contractions on L^p -spaces, Akcoglu’s pointwise maximal ergodic theorem for positive contractions on L^p -spaces, and complex interpolation of the spaces $L^p(S; \ell_n^q)$.

10.7.a The Phillips calculus

For generators of (semi)groups, another useful functional calculus, besides that of Dunford, can be defined as follows:

Definition 10.7.1 (Phillips calculus). *Let X be a Banach space and A be an operator in X .*

- (1) *If iA generates a bounded C_0 -group $(U(t))_{t \in \mathbb{R}}$ on X , we define*

$$\Phi_g(A)x := \int_{-\infty}^{\infty} g(t)U(t)x \, dt, \quad g \in L^1(\mathbb{R}), \, x \in X.$$

- (2) *If $-A$ generates a bounded C_0 -semigroup $(S(t))_{t \geq 0}$ on X , we define*

$$\Phi_g^+(A)x := \int_0^{\infty} g(t)S(t)x \, dt, \quad g \in L^1(\mathbb{R}_+), \, x \in X.$$

It is immediate that the integrals exist and

$$\begin{aligned} \|\Phi_g(A)\|_{\mathcal{L}(X)} &\leq \|g\|_{L^1(\mathbb{R})} \sup_{t \in \mathbb{R}} \|U(t)\|_{\mathcal{L}(X)}, \\ \|\Phi_g^+(A)\|_{\mathcal{L}(X)} &\leq \|g\|_{L^1(\mathbb{R}_+)} \sup_{t \in \mathbb{R}_+} \|S(t)\|_{\mathcal{L}(X)}. \end{aligned} \tag{10.43}$$

As in the case of the H^∞ calculus, the principal interest is in achieving such estimates for other function classes in place of L^1 . This question will be considered in the rest of this chapter. First, however, we would like to relate the Phillips calculus to the Dunford calculus:

Proposition 10.7.2. *Let X be a Banach space and A be an operator in X .*

(1) If iA generates a bounded C_0 -group $(U(t))_{t \in \mathbb{R}}$ on X , then

$$f(A) = \Phi_{g_f}(A) \quad \text{for all } f \in H^1(\Sigma_\sigma^{\text{bi}}), \quad 0 < \sigma < \tfrac{1}{2}\pi,$$

where

$$g_f(t) := \begin{cases} \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^-} e^{zt} f(iz) \, dz, & t > 0, \\ -\frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^+} e^{zt} f(iz) \, dz, & t < 0, \end{cases} \quad (10.44)$$

satisfies $\|g_f\|_{L^1(\mathbb{R})} \leq (\pi \sin \sigma)^{-1} \|f\|_{H^1(\Sigma_\sigma^{\text{bi}})}$.

(2) If $-A$ generates a bounded C_0 -semigroup $(S(t))_{t \in \mathbb{R}}$ on X , then

$$f(A) = \Phi_{g_f^+}(A) \quad \text{for all } f \in H^1(\Sigma_\sigma), \quad \tfrac{1}{2}\pi < \sigma < \pi,$$

where

$$g_f(t) := -\frac{1}{2\pi i} \int_{\partial \Sigma_\nu} e^{zt} f(z) \, dz, \quad t > 0, \quad (10.45)$$

satisfies $\|g_f\|_{L^1(\mathbb{R})} \leq (2\pi \sin \sigma)^{-1} \|f\|_{H^1(\Sigma_\sigma)}$.

Proof. (1): It follows from the assumption that A is bisectorial of angle 0. Hence for $0 < \sigma < \frac{1}{2}\pi$ and $f \in H^1(\Sigma_\sigma^{\text{bi}})$ the Dunford integral

$$f(A)x = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^{\text{bi}}} f(z) R(z, A)x \, dz, \quad x \in X,$$

is well defined; here $0 < \nu < \sigma$ is arbitrary and $\partial \Sigma_\nu$ is oriented in such a way that Σ_ν is on the left-hand side when traversing it.

First substituting $z = i\zeta$, then using the representation of the resolvent of a semigroup generator as the Laplace transform of the semigroup (Proposition G.4.1) and Fubini's theorem (the use of which will be justified shortly), for all $x \in X$ we obtain

$$\begin{aligned} f(A)x &= \frac{1}{2\pi} \int_{-i\partial \Sigma_\nu^{\text{bi}}} f(i\zeta) R(i\zeta, A)x \, d\zeta \\ &\stackrel{(*)}{=} -\frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^+} f(i\zeta)(\zeta + iA)^{-1} x \, d\zeta \\ &\quad - \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^-} f(i\zeta)(\zeta + iA)^{-1} x \, d\zeta \\ &\stackrel{(**)}{=} -\frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^+} f(i\zeta) \int_0^\infty e^{-\zeta t} U(-t)x \, dt \, d\zeta \\ &\quad + \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^-} f(i\zeta) \int_0^\infty e^{\zeta t} U(t)x \, dt \, d\zeta \end{aligned}$$

$$= \int_{-\infty}^{\infty} g_f(t) U(t)x \, dt = \Phi_{g_f}(A)x.$$

In (*) we used that $-i\partial\Sigma_\nu^{\text{bi}} = -(\partial\Sigma_\nu^+ + \partial\Sigma_\nu^-)$, and in (**) that $-iA$ generates the C_0 -group $(U(-t))_{t \in \mathbb{R}}$. Note that

$$\begin{aligned} \int_0^\infty |g_f(t)| \, dt &\leq \frac{1}{2\pi} \int_{\partial\Sigma_{\frac{1}{2}\pi-\nu}^-} \int_0^\infty e^{\Re z t} |f(iz)| \, dt \, |dz| \\ &= \frac{1}{2\pi} \int_{\partial\Sigma_{\frac{1}{2}\pi-\nu}^-} |f(iz)| \frac{|dz|}{|\Re z|} = \frac{1}{2\pi \sin \nu} \int_{\partial\Sigma_{\frac{1}{2}\pi-\nu}^-} |f(iz)| \frac{|dz|}{|z|} \end{aligned}$$

and likewise

$$\int_{-\infty}^0 |g_f(t)| \, dt \leq \frac{1}{2\pi \sin \nu} \int_{\partial\Sigma_{\frac{1}{2}\pi-\nu}^+} |f(iz)| \frac{|dz|}{|z|}.$$

It follows that $g_f \in L^1(\mathbb{R})$ and, letting $\nu \rightarrow \sigma$, that $\|g_f\|_1 \leq \frac{1}{\pi \sin \sigma} \|f\|_{H^1(\Sigma_\sigma^{\text{bi}})}$; at the same time this justifies the above use of Fubini's theorem.

(2): Now A is sectorial of angle $\frac{1}{2}\pi$. Proceeding as in the bisectorial case, an easy estimate shows that $g_f^+ \in L^1(\mathbb{R}_+)$ with $\|g_f^+\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{2\pi \cos \sigma} \|f\|_{H^1(\Sigma_\sigma)}$, and using Fubini's theorem with $\frac{1}{2}\pi < \nu < \sigma$ we obtain

$$\begin{aligned} \int_0^\infty g_f(t) S(t)x \, dt &= -\frac{1}{2\pi i} \int_0^\infty \int_{\partial\Sigma_\nu} e^{zt} f(z) S(t)x \, dz \, dt \\ &= -\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z) \int_0^\infty e^{zt} S(t)x \, dt \, dz \\ &= -\frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z) R(-z, -A)x \, dz \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(z) R(z, A)x \, dz = f(A)x. \end{aligned}$$

as desired. $\square$

The relation of the functions g_f and g_f^+ with f is most conveniently expressed in terms of the Fourier transform:

Lemma 10.7.3. *Under the assumptions of Proposition 10.7.2, we have*

$$\widehat{g}_f(\xi) = f(-2\pi\xi), \quad \widehat{g}_f^+(\xi) = -f(2\pi i\xi).$$

Proof. The Fourier transform of g_f is easily computed: for $\xi \neq 0$ we have

$$\widehat{g}_f(\xi) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\frac{1}{2}\pi-\nu}^-} \int_0^\infty e^{-2\pi i t \xi} e^{zt} f(iz) \, dt \, dz$$

$$\begin{aligned}
& -\frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^+} \int_{-\infty}^0 e^{-2\pi i t \xi} e^{zt} f(iz) dt dz \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^-} \frac{1}{2\pi i \xi - z} f(iz) dz + \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{1}{2}\pi-\nu}^+} \frac{1}{2\pi i \xi - z} f(iz) dz \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}^-} \frac{1}{2\pi \xi + \zeta} f(\zeta) d\zeta + \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}^+} \frac{1}{2\pi \xi + \zeta} f(\zeta) d\zeta \\
&= f(-2\pi \xi),
\end{aligned}$$

using Cauchy's theorem in the last step, noting that one integral gives $f(-2\pi \xi)$ and the other 0.

Similarly, for $\xi \neq 0$ we have

$$\begin{aligned}
\widehat{g}_f^+(\xi) &= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}} \int_0^{\infty} e^{-2\pi i t \xi} e^{zt} f(z) dt dz \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu}} \frac{1}{2\pi i \xi - z} f(z) dz \\
&= -f(2\pi i \xi)
\end{aligned}$$

by Cauchy's theorem, keeping in mind that $\frac{1}{2}\pi < \nu < \pi$ in this case. $\square$

We record an estimate that will play a role in an application of the Mihlin multiplier theorem (Corollary 8.3.11) further below.

Lemma 10.7.4. *If $f \in H^1(\Sigma_{\sigma}^{\text{bi}}) \cap H^{\infty}(\Sigma_{\sigma}^{\text{bi}})$ with $0 < \sigma < \frac{1}{2}\pi$, then the function g_f of (10.44) satisfies*

$$\sup_{\xi \in \mathbb{R} \setminus \{0\}} \max\{|\widehat{g}_f(\xi)|, |\xi \widehat{g}_f'(\xi)|\} \leq \frac{1}{\sin \sigma} \|f\|_{H^{\infty}(\Sigma_{\sigma}^{\text{bi}})}.$$

If $f \in H^1(\Sigma_{\sigma}) \cap H^{\infty}(\Sigma_{\sigma})$ with $\frac{1}{2}\pi < \sigma < \pi$, then the function g_f^+ of (10.45) satisfies

$$\sup_{\xi \in \mathbb{R} \setminus \{0\}} \max\{|\widehat{g}_f^+(\xi)|, |\xi (\widehat{g}_f^+)'(\xi)|\} \leq \frac{1}{\cos \sigma} \|f\|_{H^{\infty}(\Sigma_{\sigma}^{\text{bi}})}.$$

Proof. The claimed estimates, even with better constant 1, for $|\widehat{g}_f(\xi)|$ and $|\widehat{g}_f^+(\xi)|$ follow at once from Lemma 10.7.3.

Concerning the derivative, we can use Cauchy's theorem to write

$$|f(z)| = \left| \frac{-1}{2\pi i} \oint_{\partial D(z,r)} \frac{f(w)}{(z-w)^2} dw \right| \leq \frac{1}{2\pi} \frac{\|f\|_{\infty}}{r^2} 2\pi r = \frac{\|f\|_{\infty}}{r},$$

whenever the disc $D(z, r)$ is in the domain of f . In the first case, we are concerned with $z = -2\pi \xi \in \mathbb{R}$ and we can take $r \rightarrow |z| \sin \sigma$; in the second case, we have $z = i2\pi \xi \in i\mathbb{R}$ and we can take $r \rightarrow |z| \cos \sigma$. Noting the cancellation of the factor $2\pi|\xi|$ appearing in both the numerator and the denominator, we obtain the claimed result. $\square$

10.7.b Coifman–Weiss transference theorems

Given $g \in L^1(\mathbb{R})$, we define the convolution operator $K_g : L^p(\mathbb{R}; X) \rightarrow L^p(\mathbb{R}; X)$ for $1 \leq p < \infty$ by

$$K_g f(t) := \int_{-\infty}^{\infty} g(s) f(t-s) ds, \quad f \in L^p(\mathbb{R}; X). \quad (10.46)$$

With the help of Young's inequality one checks that

$$\|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq \|g\|_1. \quad (10.47)$$

Theorem 10.7.5 (Coifman–Weiss, transference for bounded groups).

Let iA be the generator of a uniformly bounded C_0 -group $(U(t))_{t \in \mathbb{R}}$ on a Banach space X and let $g \in L^1(\mathbb{R})$. Then for all $p \in [1, \infty)$, the Phillips calculus satisfies

$$\|\Phi_g(A)\|_{\mathcal{L}(X)} \leq \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \sup_{t \in \mathbb{R}} \|U(t)\|_{\mathcal{L}(X)}^2.$$

The point of the theorem is to “transfer” an estimate on K_g to an estimate on $\Phi_g(A)$. Of course one always has the elementary estimate in (10.43) involving $\|g\|_{L^1(\mathbb{R})}$ but the theorem often provides a better estimate in situations where we can estimate $\|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}$ in a more clever way.

Example 10.7.6. Let X be Hilbert space and $p = 2$. From Parseval's identity we obtain

$$\|K_g\|_{\mathcal{L}(L^2(\mathbb{R}; X))} = \sup_{s \in \mathbb{R}} |\hat{g}(s)|.$$

Theorem 10.7.10 will present a second example. These improvements over the first inequality in (10.43) are essential in many applications.

Proof of Theorem 10.7.5. Set $\|U\|_\infty := \sup_{t \in \mathbb{R}} \|U(t)\|$. By estimates (10.47) and (10.43), and density in $L^1(\mathbb{R})$, without loss of generality we may assume that g has support in interval $[-N, N]$, say. Given $\varepsilon > 0$, we choose M so large that $\frac{M+N}{M} \leq 1 + \varepsilon$. Since $(U(t))^{-1} = U(-t)$ we have, for all $y \in X$,

$$\|y\| \leq \|U\|_\infty \|U(-t)y\|.$$

Applying this with $y = \Phi_g(A)x$, by averaging we obtain

$$\begin{aligned} \|\Phi_g(A)x\|^p &\leq \frac{\|U\|_\infty^p}{2M} \int_{-M}^M \|U(-s)\Phi_g(A)x\|^p ds \\ &= \frac{\|U\|_\infty^p}{2M} \int_{-M}^M \left\| \int_{-\infty}^{\infty} g(t) U(t-s)x dt \right\|^p ds \end{aligned}$$

by the definition of $\Phi_g(A)$. Also $\mathbf{1}_{[-M-N, M+N]}(t-s) = 1$ if $t \in \text{supp } g \subseteq [-N, N]$ and $s \in [-M, M]$, so that with $\chi := \mathbf{1}_{[-M-N, M+N]}$ the last expression can be rewritten as

$$= \frac{\|U\|_\infty^p}{2M} \int_{-M}^M \left\| \int_{-\infty}^\infty g(t) [U(t-s)x \chi(t-s)] dt \right\|^p ds$$

and, by the definition of K_g ,

$$\begin{aligned} &= \frac{\|U\|_\infty^p}{2M} \int_{-M}^M \|K_g[U(\cdot)x \chi(\cdot)](s)\|^p ds \\ &\leq \frac{\|U\|_\infty^p}{2M} \|K_g[U(\cdot)x \chi(\cdot)]\|_{L^p(\mathbb{R}; X)}^p \\ &\leq \frac{\|U\|_\infty^p}{2M} \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}^p \int_{-\infty}^\infty \|U(s)x \chi(s)\|^p ds \\ &= \frac{\|U\|_\infty^p}{2M} \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}^p \int_{-N-M}^{N+M} \|U(s)x\|^p ds \\ &\leq \frac{\|U\|_\infty^p}{2M} \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}^p 2(N+M) \|U\|_\infty^p \|x\|^p \\ &\leq (1+\varepsilon) \|U\|_\infty^{2p} \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}^p \|x\|^p \end{aligned}$$

and the claim follows. $\square$

Remark 10.7.7. In exactly the same way one proves the following discrete version of the theorem which will be needed shortly: if U is an invertible operator on a Banach space X satisfying $\sup_{n \in \mathbb{Z}} \|U^n\| < \infty$, then for all $a \in \ell^1(\mathbb{Z})$ we have

$$\left\| \sum_{n \in \mathbb{Z}} a_n U^n \right\|_{\mathcal{L}(X)} \leq \|\kappa_a\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))} \sup_{n \in \mathbb{Z}} \|U^n\|^2,$$

where κ_a is the convolution operator on $\ell_\mathbb{Z}^p$ given by $(\kappa_a c)_n := \sum_{k \in \mathbb{Z}} a_k c_{n-k}$.

Coifman–Weiss transference for positive contractions

In the rest of this subsection, we consider another version of the Coifman–Weiss transference. This will require some preparation.

Given a sequence $a = (a_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, we denote by κ_a the corresponding convolution operator on $\ell^p(\mathbb{Z}; X)$, $1 \leq p \leq \infty$, i.e., for $c = (c_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z}; X)$ we set

$$(\kappa_a c)_n := \sum_{k \in \mathbb{Z}} a_k c_{n-k}.$$

By Young's inequality, $\|\kappa_a\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))} \leq \|a\|_{\ell^1(\mathbb{Z})}$. In what follows, sequences $(a_n)_{n \geq 0} \in \ell^1(\mathbb{N})$ will be identified with sequences $(a_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ in the natural way by taking $a_{-n} = 0$ for $n \geq 1$.

Proposition 10.7.8. *Let $(S, \mathcal{A}, \mu)$ be a measure space, $p \in (1, \infty)$, and $T \in \mathcal{L}(L^p(S))$ be a positive contraction. Then for every $a \in \ell^1(\mathbb{N})$ and every Banach space X we have*

$$\left\| \sum_{n \geq 0} a_n T^n \right\|_{\mathcal{L}(L^p(S; X))} \leq \|\kappa_a\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))}.$$

Proof. By Theorem 2.1.3, T extends to a contraction on $L^p(S; X)$. Since both sides of the bound are dominated by $\|a\|_{\ell^1}$, by approximation, in the rest of the proof we may assume that a is finitely supported.

Step 1 – First we consider the situation that $(S, \mathcal{A}, \mu)$ is a measure space having a finite number of points and μ gives mass 1 to every point. Then $L^p(S)$ may be identified with ℓ_n^p , where n is the number of points. By the Akcoglu–Sucheston Dilation Theorem I.1.2, there is a positive, invertible, isometric $U \in L^p(D)$, for some measure space D , as well as a positive isometric embedding $J : \ell_n^p \rightarrow L^p(D)$ and a positive surjective contraction $P : L^p(D) \rightarrow \mathcal{R}(J)$ such that

$$T^k = J^{-1} P U^k J \quad \text{for all } k = 0, 1, 2, \dots$$

Since all operators are positive, this identity extends to $\ell_n^p(X)$. Hence

$$\begin{aligned} \left\| \sum_{k \geq 0} a_k T^k \right\|_{\mathcal{L}(\ell_n^p(X))} &= \left\| J^{-1} P \sum_{k \geq 0} a_k U^k J \right\|_{\mathcal{L}(\ell_n^p(X))} \\ &\leq \|J^{-1} P\|_{\mathcal{L}(L^p(D; X), \ell_n^p(X))} \left\| \sum_{k \geq 0} a_k U^k \right\|_{\mathcal{L}(L^p(D; X))} \|J\|_{\mathcal{L}(\ell_n^p(X), L^p(D; X))} \\ &\leq 1 \cdot \|\kappa_a\|_{\mathcal{L}(\ell^p(\mathbb{Z}))} \cdot 1, \end{aligned}$$

where the last step used the following results: For the middle factor, this was an application of Theorem 10.7.5 and Remark 10.7.7. For the other two factors, the bounds are guaranteed by Theorem I.1.2 when $X = \mathbb{K}$, and they follow from this and the positivity for general X via Theorem 2.1.3.

Step 2 – We suppose again that $(S, \mathcal{A}, \mu)$ consists of finitely many points, but drop the assumption that μ assigns equal mass to each of them. After deleting the points of zero mass, we may assume that all points have strictly positive mass. Denoting by ν the measure that gives mass one to each point, we obtain an isometric isomorphism $L^p(S, \mu; X) \simeq L^p(S, \nu; X)$ by sending $f \in L^p(S, \mu; X)$ to the function $\tilde{f} \in L^p(S, \nu; X)$ given by $\tilde{f}(s) := \mu(\{s\})^{1/p} f(s)$. Along this isomorphism, T can be identified with a positive contraction $\tilde{T}$ on $L^p(S, \nu)$. The result now follows from Step 1.

Step 3 – Now let $(S, \mathcal{A}, \mu)$ be an arbitrary measure space and fix a function $f \in L^p(S; X)$. Since each of the functions $T^n f$ can be approximated by μ -simple functions, in proving the estimate

$$\left\| \sum_{n \geq 0} a_n T^n f \right\| \leq \|\kappa_a\|_{\ell^p(\mathbb{Z})} \|f\|$$

we may replace the σ -algebra $\mathcal{A}$ by the σ -algebra generated by the countably many sets of finite μ -measure involved in the above-mentioned approximation of the functions $T^n f$, $n \geq 0$. Let $(A_j)_{j \geq 1}$ be an enumeration of these sets.

Thus, we may assume that $\mu(A_j) < \infty$ for all $j \geq 1$ and that $\mathcal{A}$ is generated by the sets A_j , $j \geq 1$.

For each $k \geq 1$, let $\mathbb{E}_k$ denote the conditional expectation in $L^p(S)$ associated with the finite σ -algebra $\mathcal{A}_k$ generated by the sets $A_1, \dots, A_k$. The operator $T_k := \mathbb{E}_k \circ T$ restricts to a positive contraction on $L^p(S, \mathcal{A}_k)$. Up to a set of μ -measure zero, every set in the finite σ -algebra $\mathcal{A}_k$ is a finite union of atoms of finite and strictly positive μ -measure, and therefore Step 2 gives us the inequality

$$\left\| \sum_{n \geq 0} a_n T_k^n f \right\|_{L^p(S; X)} \leq \|\kappa_a\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))} \|f\|_{L^p(S; X)}, \quad k = 1, 2, \dots$$

The infinite sum on the left-hand side effectively extends over the finite set of indices for which a_n is non-zero. Since $T_k f \rightarrow T f$ in $L^p(S, \mathcal{A})$ strongly (recalling that we replaced the original σ -algebra by the σ -algebra generated by the sets A_j , $j \geq 1$), upon passing to the limit $k \rightarrow \infty$ we obtain the desired estimate. $\square$

By using Proposition 10.7.8, it is possible to prove an analogue of Theorem 10.7.5 for generators of positive C_0 -contraction semigroups on $L^p(S)$. We recall from (10.46) the notation K_g for the convolution operator. Functions on $\mathbb{R}_+$ will be identified with function on $\mathbb{R}$ in the natural way, i.e., by extending them by zero on the negative axis.

Theorem 10.7.9 (Coifman–Weiss, transference for positive contraction semigroups on L^p -spaces). *Let $(S, \mathcal{A}, \mu)$ be a measure space, let $1 \leq p < \infty$, and let X be a Banach space. Let $-A$ be the generator of a positive C_0 -semigroup $(T(t))_{t \geq 0}$ of contractions on $L^p(S)$, and hence on $L^p(S; X)$. Then the Phillips calculus satisfies, for all $g \in L^1(\mathbb{R}_+)$,*

$$\|\Phi_g^+(A)\|_{\mathcal{L}(L^p(S; X))} \leq \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}.$$

Proof. As in the discrete case, Young's inequality allows us to reduce the proof of the theorem to the case where g is a step function supported on finitely many dyadic intervals $[j2^{-n}, (j+1)2^{-n})$. The idea of the proof is to also approximate the semigroup by powers of a fixed operator and then to apply the bound for the discrete case.

Consider, for $N = 1, 2, \dots$ and $j \in \mathbb{Z}$, the numbers

$$a_j^{(N)} := \frac{1}{2^N} \int_0^1 \int_0^1 g\left(\frac{j+t-s}{2^N}\right) ds dt, \quad (10.48)$$

where we think g as a function in $L^p(\mathbb{R}; X)$ by zero-extension on $\mathbb{R}_-$. Identifying the finite sequence $a^{(N)} = (a_j^{(N)})_{j \in \mathbb{Z}}$ with an element of $\ell^1(\mathbb{Z})$, for a finitely non-zero sequence $c = (c_j)_{j \in \mathbb{Z}}$ we have

$$(\kappa_{a^{(N)}} c)_k = \sum_{j \in \mathbb{Z}} a_{k-j}^{(N)} c_j$$

$$\begin{aligned}
&= \frac{1}{2^N} \int_0^1 \int_0^1 \sum_{j \in \mathbb{Z}} g\left(\frac{k-j+t-s}{2^N}\right) ds dt c_j \\
&= \int_k^{k+1} \sum_{j \in \mathbb{Z}} \int_j^{j+1} g^{(N)}(t-s) ds dt c_j \\
&\quad \text{(setting } g^{(N)}(u) := 2^{-N} g(2^{-N}u)) \\
&= \int_k^{k+1} \int_{-\infty}^{\infty} g^{(N)}(t-s) C(s) ds dt \\
&\quad \text{(setting } C := \sum_{j \in \mathbb{Z}} c_j \mathbf{1}_{(j, j+1)}) \\
&= E_k(g^{(N)} * C) \quad \text{(setting } E_k h := \int_k^{k+1} h(u) du).
\end{aligned}$$

Let $\mathbb{E}_{\mathbb{Z}}$ denote the conditional expectation with respect to the σ -algebra generated by the intervals $[k, k+1)$. Then

$$\begin{aligned}
\|\kappa_{a^{(N)}} c\|_{\ell^p(\mathbb{Z}; X)} &= \left(\sum_{k \in \mathbb{Z}} \|E_k(g^{(N)} * C)\|_X^p \right)^{1/p} \\
&= \|\mathbb{E}_{\mathbb{Z}}(g^{(N)} * C)\|_{L^p(\mathbb{R}; X)} \leq \|g^{(N)} * C\|_{L^p(\mathbb{R}; X)} \\
&\leq \|K_{g^{(N)}}\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \|C\|_{L^p(\mathbb{R}; X)} = \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \|c\|_{\ell^p(\mathbb{Z})},
\end{aligned}$$

where the last identity follows by routine changes of variables.

So far, the sequence $c = (c_j)_{j \in \mathbb{Z}}$ was finitely non-zero. Since such sequences are dense in $\ell^p(\mathbb{Z}; X)$, the above estimates prove that

$$\|\kappa_{a^{(N)}}\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))} \leq \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \quad \text{for all } N.$$

Now we are in a position to apply Proposition 10.7.8 to the operator $T := T(2^{-N})$ and obtain the estimate

$$\begin{aligned}
\left\| \sum_{j \geq 0} a_j^{(N)} T(j2^{-N}) \right\|_{\mathcal{L}(L^p(S; X))} &\leq \|\kappa_{a^{(N)}}\|_{\mathcal{L}(\ell^p(\mathbb{Z}; X))} \\
&\leq \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}; X))}.
\end{aligned} \tag{10.49}$$

The final step is to realise that, in a sense to be made precise, $a_j^{(N)} \approx 2^{-N} g(j2^{-N})$ so that

$$\sum_{j \geq 0} a_j^{(N)} T(j2^{-N}) f \approx 2^{-N} \sum_{j \geq 0} g(j2^{-N}) T(j2^{-N}) f.$$

The latter is a Riemann sum for the integral $\Phi_g(A)f = \int_0^\infty g(t)T(t)f dt$.

Turning to the details, we begin by recalling the assumption, made at the beginning of the proof, that g is a dyadic step function, say supported on finitely many dyadic intervals of length 2^{-n} , where n is some given positive integer. Then for $N \geq n$ we have $g(t) = g(j2^{-N})$ whenever t belongs to a

dyadic interval of the form $[j2^{-N}, (j+1)2^{-N})$. As a result it is immediately obvious that

$$\|\Phi_g(A)f\|_{L^p(S;X)} \leq \liminf_{N \rightarrow \infty} \left\| 2^{-N} \sum_{j \geq 0} g(j2^{-N})T(j2^{-N})f \right\|_{L^p(S;X)},$$

keeping in mind that the sum on the right-hand side effectively extends over at most finitely many non-zero terms. Again since g is dyadic, for $N \geq n$ the definition (10.48) implies that $a_j^{(N)} = 2^{-N}g(j2^{-N})$ for all $0 \leq j \leq 2^N - 1$ except possibly for values of the form $j = k2^{N-n}$. Therefore

$$\begin{aligned} \liminf_{N \rightarrow \infty} \left\| \sum_{j \geq 0} a_j^{(N)} T(j2^{-N})f \right\|_{L^p(S;X)} \\ = \liminf_{N \rightarrow \infty} \left\| 2^{-N} \sum_{j \geq 0} g(j2^{-N})T(j2^{-N})f \right\|_{L^p(S;X)}, \end{aligned} \quad (10.50)$$

as the relative contribution of the exceptional j 's tends to zero as $N \rightarrow \infty$.

The proof is concluded by combining the estimates (10.49) and (10.50). $\square$

10.7.c Hieber–Prüss theorems on H^∞ -calculus for generators

We are now in a position to present neat corollaries of the preceding considerations in the context of H^∞ -calculus. Recalling the notion of Fourier multipliers from Section 8.3, we observe that

$$\mathcal{F}(K_g f) = \mathcal{F}(g * f) = \widehat{g\hat{f}} = \mathcal{F}(T_{\widehat{g}} f), \quad (10.51)$$

so that K_g coincides with the Fourier multiplier $T_{\widehat{g}}$.

Theorem 10.7.10 (Hieber–Prüss I). *Let iA be the generator of a uniformly bounded C_0 -group $(U(t))_{t \in \mathbb{R}}$ on a UMD space X . Then A has a bounded bisectorial H^∞ -calculus of angle 0.*

Proof. Recall from Example 10.6.3 that if iA is the generator of a uniformly bounded C_0 -group, then A is bisectorial of angle 0.

By Proposition 10.7.2 and Theorem 10.7.5, we have

$$\|f(A)\|_{\mathcal{L}(X)} = \|\Phi_{g_f}(A)x\|_{\mathcal{L}(X)} \leq \|K_{g_f}\|_{\mathcal{L}(L^p(\mathbb{R};X))} \|U\|_\infty^2.$$

With observation (10.51), the Mihlin multiplier theorem (Corollary 8.3.11) and Lemma 10.7.4, we obtain

$$\begin{aligned} \|K_{g_f}\|_{\mathcal{L}(L^p(\mathbb{R};X))} &= \|T_{\widehat{g_f}}\|_{\mathcal{L}(L^p(\mathbb{R};X))} \\ &\leq 200\beta_{p,X}^2 h_{p,X} \sup_{\xi \in \mathbb{R} \setminus \{0\}} \max\{|\widehat{g_f}(\xi)|, |\xi \widehat{g_f}'(\xi)|\} \\ &\leq 200\|U\|_\infty^2 \beta_{p,X}^2 h_{p,X} \frac{1}{\sin \sigma} \|f\|_{H^\infty(\Sigma_\sigma^{\text{bi}})}. \end{aligned}$$

This proves that A has a bounded bisectorial $H^\infty(\Sigma_\sigma^{\text{bi}})$ -calculus. Since the choice of $0 < \sigma < \frac{1}{2}\pi$ was arbitrary, the theorem is proved. $\square$

Corollary 10.7.11. *Let iA generate a bounded C_0 -group on a UMD Banach space X . If A is R -bisectorial, then A^2 is R -sectorial and has a bounded H^∞ -calculus of angle 0.*

Proof. Combine the previous theorem with Theorem 10.6.7. $\square$

Theorem 10.7.12 (Hieber–Prüss II). *Let $(S, \mathcal{A}, \mu)$ be a measure space, $1 < p < \infty$, and X be a UMD space. Let $-A$ be the generator of a positive C_0 -contraction semigroup $(S(t))_{t \geq 0}$ on $L^p(S)$, and thus on $L^p(S; X)$. Then A has a bounded H^∞ -calculus on $L^p(S; X)$ of angle $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$.*

Proof. By Proposition 10.7.2 and Theorem 10.7.9, for functions $f \in H^1(\Sigma_\sigma)$ with $\frac{1}{2}\pi < \sigma < \pi$ we obtain

$$\|f(A)\|_{\mathcal{L}(X)} = \|\Phi_{g_f^+}^+(A)\|_{\mathcal{L}(X)} \leq \|K_{g_f^+}\|_{L^p(\mathbb{R}; X)},$$

where $K_{g_f^+}$ denotes convolution with g_f^+ .

By observation (10.51), the Mihlin multiplier theorem (Corollary 8.3.11) and Lemma 10.7.4, we obtain

$$\begin{aligned} \|K_{g_f^+}\|_{\mathcal{L}(L^p(\mathbb{R}; X))} &= \|T_{\widehat{g_f^+}}\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \\ &\leq 200\beta_{p,X}^2 h_{p,X} \sup_{\xi \in \mathbb{R} \setminus \{0\}} \max\{|\widehat{g_f^+}(\xi)|, |\xi(\widehat{g_f^+})'(\xi)|\} \\ &\leq 200\|U\|_\infty^2 \beta_{p,X}^2 h_{p,X} \frac{1}{\cos \sigma} \|f\|_{H^\infty(\Sigma_\sigma)}. \end{aligned}$$

$\square$

10.7.d Analytic semigroups of positive contractions on L^p

In Theorem 10.7.12 we have seen that if $-A$ generates a positive C_0 -contraction semigroup $(S(t))_{t \geq 0}$ on a space $L^p(S)$ with $1 < p < \infty$, then A admits a bounded H^∞ -calculus and $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$. The following theorem shows that under stronger assumptions one obtains $\omega_{H^\infty}(A) < \frac{1}{2}\pi$. Unlike Theorem 10.7.12, the one that follows is restricted to the scalar-valued case.

Theorem 10.7.13. *Let $-A$ be the generator of a positive C_0 -contraction semigroup $(S(t))_{t \geq 0}$ on a space $L^p(S)$, where $(S, \mathcal{A}, \mu)$ is a measure space and $1 < p < \infty$. If $(S(t))_{t \geq 0}$ extends to a bounded analytic C_0 -contraction semigroup on the sector Σ_η for some $0 < \eta < \frac{1}{2}\pi$, then A is R -sectorial, admits a bounded H^∞ -calculus, and we have*

$$\omega_R(A) = \omega_{H^\infty}(A) \leq \frac{1}{2}\pi - \frac{1}{2}\eta(p \wedge p').$$

To prepare for the proof, we begin by preparing some lemmas for bounded analytic operator-valued functions $z \mapsto N(z)$ defined on a sector. Eventually we shall apply these results to the analytic extension of the function N given by

$$N(t)f := \frac{1}{t} \int_0^t S(s)f \, ds, \quad f \in L^p(S).$$

Our main tool for estimating N will be the following semigroup version of Akcoglu's maximal ergodic theorem:

Theorem 10.7.14 (Akcoglu's maximal ergodic theorem). *If the C_0 -semigroup $(S(t))_{t \geq 0}$ is positive and contractive on $L^p(S)$, with $1 < p < \infty$, then for all $f \in L^p(S)$ we have*

$$\left\| \sup_{t > 0} \left| \frac{1}{t} \int_0^t S(s)f \, ds \right| \right\|_p \leq p' \|f\|_p.$$

Proof. The main step is to prove that for all choices of finitely many dyadic $t_1, \dots, t_k > 0$ and for all $f \in L^p(S)$ we have

$$\left\| \sup_{1 \leq j \leq k} \left| \frac{1}{t_j} \int_0^{t_j} S(s)f \, ds \right| \right\|_p \leq p' \|f\|_p.$$

Once we have this, the general case follows by considering an enumeration of the positive dyadic numbers, say $(q_n)_{n \geq 1}$, applying the above with $t_j = q_j$ for $j = 1, \dots, k$, and passing to the limit $k \rightarrow \infty$. This will give the result with ' $t > 0$ ' replaced by ' $t > 0$ and dyadic'; as a last step we can extend to supremum to all $t > 0$ by the following limiting argument. From

$$\left\| \int_0^t S(s)f \, ds \right\|_p \leq \int_0^t \|S(s)f\|_p \, ds$$

and the boundedness of the semigroup it is immediate that $s \mapsto (S(s)f)(x)$ is locally integrable at almost every $x \in S$, and then $t \mapsto \frac{1}{t} \int_0^t S(s)f(x) \, ds$ is continuous at the same values of x . From this the approximation by dyadic values is immediate.

Let us now fix dyadic numbers $t_1, \dots, t_k > 0$. Fix an arbitrary $\varepsilon > 0$ and choose the integer $N \geq 0$ so large that

$$\|S(t)f - f\| < \varepsilon \text{ for all } t \in [0, 2^{-N}]. \quad (10.52)$$

We may assume that the numbers t_j are pairwise different dyadic numbers and have the form $t_j = m_j 2^{-n}$ with $n \geq N$. It then suffices to prove the estimate

$$\left\| \sup_{1 \leq i \leq M} \left| \frac{1}{i 2^{-n}} \int_0^{i 2^{-n}} S(s)f \, ds \right| \right\|_p \leq \|f\|_p.$$

Upon replacing f by $|f|$, there is no loss of generality if we assume $f \geq 0$. We have, for $n \geq N$,

$$\begin{aligned} & \left\| \sup_{1 \leq i \leq M} \frac{1}{i2^{-n}} \int_0^{i2^{-n}} S(s)f \, ds \right\|_p \\ &= \left\| \sup_{1 \leq i \leq M} \frac{1}{i2^{-n}} \sum_{\ell=0}^{i-1} \int_{\ell2^{-n}}^{(\ell+1)2^{-n}} S(s)f \, ds \right\|_p \\ &= \left\| \sup_{1 \leq i \leq M} \frac{1}{i} \sum_{\ell=0}^{i-1} S(\ell2^{-n}) \tilde{S}_{2^{-n}} f \right\|_p \quad (\text{with } \tilde{S}_{2^{-n}} f := \frac{1}{2^{-n}} \int_0^{2^{-n}} S(s)f \, ds) \\ &\leq p' \|\tilde{S}_{2^{-n}} f\|_p \leq p'(\|f\|_p + \varepsilon), \end{aligned}$$

where the first estimate was an application of Akcoglu's maximal ergodic theorem in its discrete form given in Theorem 1.2.1, and the last step used (10.52). Since $\varepsilon > 0$ was arbitrary, this completes the proof. $\square$

Corollary 10.7.15. *If the C_0 -semigroup $(S(t))_{t \geq 0}$ is positive and contractive on $L^p(S)$, with $1 < p < \infty$, then for all $t > 0$ and $f_1, \dots, f_k \in L^p(S)$ we have*

$$\left\| \sup_{1 \leq j \leq k} \sup_{t > 0} \left| \frac{1}{t} \int_0^t S(s)f_j \, ds \right| \right\|_p \leq p' \left\| \sup_{1 \leq j \leq k} |f_j| \right\|_p.$$

Proof. Apply the theorem with $f := \sup_{1 \leq j \leq k} |f_j|$. $\square$

Lemma 10.7.16. *Let $p \in [1, 2]$ and suppose that $N : \overline{\Sigma_\delta} \rightarrow \mathcal{L}(L^p(S))$ is a strongly continuous and uniformly bounded function, which is holomorphic on Σ_δ . If there is a constant $C \geq 0$ such that, for all choices of finitely many $t_1, \dots, t_k \in \mathbb{R}_+$ and $f_1, \dots, f_k \in L^p(S)$,*

$$\left\| \sup_{1 \leq j \leq k} |N(t_j)f_j| \right\|_p \leq C \left\| \sup_{1 \leq j \leq k} |f_j| \right\|_p, \quad (10.53)$$

then the family $\mathcal{T} = \{N(z) : z \in \Sigma_{\delta p/2}\}$ is R -bounded and

$$\mathcal{R}(\mathcal{T}) \leq \kappa_{2,p}^2 C^{1-p/2} \sup_{t > 0} \|N(t)\|_p^{p/2}.$$

Proof. For $p = 2$ this is clear. We may therefore assume that $p \in [1, 2)$.

The boundedness of $N(z)$ on $\overline{\Sigma_\delta}$ implies that for $z_1, \dots, z_k$ with $|\arg z_j| = \delta$ and $f_1, \dots, f_k \in L^p(S)$ we have

$$\begin{aligned} & \left\| \left(\sum_{j=1}^k |N(z_j)f_j|^p \right)^{1/p} \right\|_p^p = \sum_{j=1}^k \|N(z_j)f_j\|_p^p \\ & \leq K^p \sum_{j=1}^k \|f_j\|_p^p = K^p \left\| \left(\sum_{j=1}^k |f_j|^p \right)^{1/p} \right\|_p^p, \end{aligned} \quad (10.54)$$

where $K := \sup_{z \in \overline{\Sigma_\delta}} \|N(z)\|$. On the other hand, to prove the lemma it suffices to show the following square function estimate: for all $z_1, \dots, z_k$ with $|\arg z_j| = \delta p/2$ and $f_1, \dots, f_k \in L^p(S)$,

$$\left\| \left(\sum_{j=1}^k |N(z_j) f_j|^2 \right)^{1/2} \right\|_p \leq C^{1-p/2} K^{p/2} \left\| \left(\sum_{j=1}^k |f_j|^2 \right)^{1/2} \right\|_p. \quad (10.55)$$

Indeed, up to a multiplicative constant $\kappa_{2,p} \kappa_{p,2} = \kappa_{2,p}$ in both directions, by Proposition 6.3.3 the square functions on both sides of (10.55) are equivalent to Rademacher sums, and interpreted in this way the inequality asserts the R -boundedness of the family $\{N(z) : |\arg z| = \delta p/2\}$ with R -bound at most $\kappa_{2,p}^2 C$. Then $\{N(z) : |\arg z| \leq \delta p/2\}$ is R -bounded, with the same R -bound, by Proposition 8.5.8.

We shall obtain the estimate (10.55) (which can be interpreted as ℓ^2 -boundedness of the operators $N(t)$) by interpolating the inequalities (10.53) and (10.54) (which can be interpreted as ℓ^∞ -boundedness and ℓ^p -boundedness of these operators). To make this precise we fix $t_1, \dots, t_k \in \mathbb{R}_+$ and define a strongly continuous mapping $M : \overline{\Sigma_\delta} \rightarrow \mathcal{L}(L^p(S, \ell_k^p))$ by

$$M(z)(f_1, \dots, f_k) = (N(t_1 z) f_1, \dots, N(t_k z) f_k).$$

This mapping is holomorphic on Σ_δ . Now (10.53) says that

$$\|M(z)\|_{\mathcal{L}(L^p(S, \ell_k^\infty))} \leq C \text{ for } \arg z = 0$$

and (10.54) says that

$$\|M(z)\|_{\mathcal{L}(L^p(S, \ell_k^p))} \leq K \text{ for } |\arg z| = \delta.$$

Since $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$ for $\theta = \frac{p}{2}$, the interpolation lemma 10.7.17 below (with $\sigma_0 = 0$, $\sigma_1 = \delta$, $p_0 = p_1 = p$, $q_0 = \infty$, $q_1 = 2$, and $\theta = p/2$) gives

$$\|M(z)\|_{\mathcal{L}(L^p(S, \ell_k^2))} \leq C^{1-p/2} K^{p/2} \text{ for } |\arg z| = \delta p/2,$$

which in turn implies (10.55) for $z_j = t_j e^{\pm i \delta p/2}$. $\square$

In the next lemma we use the notation $\Sigma_{\sigma_0, \sigma_1} = \{z \in \mathbb{C} : \sigma_0 < \arg z < \sigma_1\}$.

Lemma 10.7.17. *Let $p_0, p_1, q_0, q_1 \leq \infty$ and fix an integer $k \geq 1$. Consider a family of operators*

$$N(z) : L^{p_0}(S, \ell_k^{q_0}) \cap L^{p_1}(S, \ell_k^{q_1}) \rightarrow L^{p_0}(S, \ell_k^{q_0}) + L^{p_1}(S, \ell_k^{q_1})$$

satisfying the following conditions:

- *as an $L^{p_0}(S; \ell_k^{q_0}) + L^{p_1}(S; \ell_k^{q_1})$ -valued function, $z \rightarrow N(z)F$ is continuous on $\overline{\Sigma_{\sigma_0, \sigma_1}}$ and holomorphic on $\Sigma_{\sigma_0, \sigma_1}$ for all $F \in L^{p_0}(S; \ell_k^{q_0}) \cap L^{p_1}(S; \ell_k^{q_1})$;*

- the boundary functions $r \mapsto N(re^{i\sigma_j})F$ are continuous $L^{p_j}(S; \ell_k^{q_j})$ -valued functions for all $F \in L^{p_j}(S; \ell_k^{q_j})$ and

$$\|N(z)\|_{\mathcal{L}(L^{p_j}(S; \ell_k^{q_j}))} \leq L_j \quad \text{for } \arg z = \sigma_j \quad (j = 0, 1). \quad (10.56)$$

For $\theta \in (0, 1)$ put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1.$$

Then $N(z)$ maps $L^p(S; \ell_k^q)$ into itself for all $\arg(z) = \sigma$ and

$$\|N(z)\|_{\mathcal{L}(L^p(S; \ell_k^q))} \leq K_0^{1-\theta} K_1^\theta \quad \text{for } \arg(z) = \sigma.$$

Proof. Let $\mathbb{S} = \{z : 0 < \Re z < 1\}$ denote the unit strip and put

$$M(\zeta) := N(\exp[i(\sigma_0 + (\sigma_1 - \sigma_0)\zeta)]), \quad \zeta \in \mathbb{S}.$$

Then, by (10.56),

$$\|M(\zeta)\|_{\mathcal{L}(L^{p_j}(S; \ell_k^{q_j}))} \leq L_j \quad \text{for } \Re \zeta = j \quad (j = 0, 1).$$

By Theorem 2.2.6, for the complex interpolation method we have an isometric isomorphism

$$L^p(S, \ell_k^q) = [L^{p_0}(S, \ell_k^{q_0}), L^{p_1}(S, \ell_k^{q_1})]_\theta.$$

For $F \in L^p(S, \ell_k^q)$ with $\|F\| < 1$, by the definition of the complex method there is a continuous function $f : \overline{\mathbb{S}} \rightarrow L^{p_0}(S; \ell_k^{q_0}) + L^{p_1}(S; \ell_k^{q_1})$, holomorphic on $\mathbb{S}$, with $f(\theta) = F$ and

$$\|f(\zeta)\|_{L^{p_j}(S; \ell_k^{q_j})} \leq 1 \quad \text{for } \Re \zeta = j \quad (j = 0, 1).$$

For $z \in \Sigma_{\sigma_0, \sigma_1}$ let $\zeta(z) \in \mathbb{S}$ be the unique point such that $\exp[i(\sigma_0 + (\sigma_1 - \sigma_0)\zeta(z))] = z$.

Then for $z \in \Sigma_{\sigma_0, \sigma_1}$ with $\arg(z) = \sigma$, say $z = re^{i\sigma}$, we have $N(re^{i\sigma})F = M(\theta + i \log r)f(\theta)$. This suggests to consider the function

$$\psi(\zeta) := M(\zeta + i \log r)f(\zeta), \quad \text{for } \zeta \in \overline{\mathbb{S}}.$$

As a function with values in $L^{p_0}(S; \ell_k^{q_0}) + L^{p_1}(S; \ell_k^{q_1})$, this function is continuous on $\overline{\mathbb{S}}$ and holomorphic on $\mathbb{S}$, it satisfies $\|\psi(z)\|_{L^{p_j}(S; \ell_k^{q_j})} \leq L_j \quad (j = 0, 1)$, and we have $\psi(\theta) = N(re^{i\sigma})F$. Again by the definition of the complex method this implies $N(re^{i\sigma})F \in [L^{p_0}(S, \ell_k^{q_0}), L^{p_1}(S, \ell_k^{q_1})]_\theta = L^p(S, \ell_k^q)$ and

$$\|N(re^{i\sigma})F\|_{L^p(S; \ell_k^q)} \leq L_0^{1-\theta} L_1^\theta.$$

Since $F \in L^p(S; \ell_k^q)$ with $\|F\| < 1$ was arbitrary, this proves the lemma. $\square$

Lemma 10.7.18. *Let X and Y be Banach spaces and let $0 < \nu < \sigma < \vartheta < \pi$. Let $N : \Sigma_\vartheta \rightarrow \mathcal{L}(X, Y)$ be a bounded holomorphic function and suppose that the set $\{N(z) : z \in \partial\Sigma_\sigma, z \neq 0\}$ is R -bounded. Then*

$$\left\{ z \frac{d}{dz} N(z) : z \in \Sigma_\nu \right\}$$

is R -bounded.

Proof. By Theorem 8.5.2, this follows from Cauchy's formula

$$z \frac{d}{dz} N(z) = \int_{\partial\Sigma_\sigma} h_z(\zeta) N(\zeta) d\zeta, \quad z \in \Sigma_\nu,$$

with $h_z(\zeta) = \frac{1}{2\pi i} z(\zeta - z)^{-2}$, noting that $\sup\{\|h_z\|_1 : z \in \partial\Sigma_\nu\} < \infty$. $\square$

We are now ready to prove Theorem 10.7.13.

Proof of Theorem 10.7.13. First let $p \in (1, 2]$, and assume that the semigroup $(S(t))_{t \geq 0}$ extends analytically to a bounded semigroup $(S(z))_{z \in \Sigma_\eta}$ for some $0 < \eta < \frac{1}{2}\pi$. Then we may also extend $N(t) = \frac{1}{t} \int_0^t S(s) ds$ holomorphically to Σ_η by

$$N(z)f = \frac{1}{z} \int_{\Gamma_{0,z}} S(\mu) f d\mu, \quad z \in \Sigma_\eta,$$

where $\Gamma_{0,z}$ is the line segment connecting 0 and z . Fix $0 < \delta < \eta$ and observe that the maximal ergodic estimate of Corollary 10.7.15, namely

$$\left\| \sup_{1 \leq j \leq k} |N(t_j) f_j| \right\|_p \leq p' \left\| \sup_{1 \leq j \leq k} |f_j| \right\|_p,$$

implies (10.53) for the holomorphic function $N(z)$, with constant $C = p'$. By Lemma 10.7.16 we conclude that for $\delta' = \delta p/2$ the set $\{N(z) : z \in \Sigma_{\delta'}\}$ is R -bounded. Also, fixing any $0 < \delta'' < \delta'$, Lemma 10.7.18 implies the R -boundedness of $\{z \frac{d}{dz} N(z) : z \in \Sigma_{\delta''}\}$. Since $S(z) = N(z) + z \frac{d}{dz} N(z)$, it follows that the set $\{S(z) : z \in \Sigma_{\delta''}\}$ is R -bounded.

If $p \in (2, \infty)$, by what we just proved the adjoint family $\{S^*(z) : z \in \Sigma_{\delta''}\}$, which is a bounded analytic C_0 -semigroup on $L^{p'}(S)$, is R -bounded on $L^{p'}(S)$. By Proposition 8.4.1 (which can be applied because $L^p(S)$ is K -convex), the family $\{S(z) : z \in \Sigma_{\delta''}\}$ is R -bounded.

An appeal to Proposition 10.3.3 now shows that A is R -sectorial of angle $\omega_R(A) \leq \frac{1}{2}\pi - \delta''$. This was the heart of the matter: now that we have secured R -sectoriality we may apply Proposition 10.4.10 to conclude that $\omega_{H^\infty}(A) = \omega_R(A) \leq \frac{1}{2}\pi - \delta''$ (the proposition can be applied since we already know, by virtue of Theorem 10.7.12, that A has a bounded H^∞ -calculus (of angle $\leq \frac{1}{2}\pi$)).

Tracing the optimal values of the various angles provided by the above proof, we find the explicit bound

$$\omega_{H^\infty}(A) \leq \frac{1}{2}\pi - \frac{1}{2}\eta(p \wedge p').$$

This completes the proof of the theorem. $\square$

10.8 Notes

The H^∞ -calculus for sectorial operators on Hilbert spaces was developed by [McIntosh \[1986\]](#) as a tool in his pursuit of the Kato square root problem. His work introduced harmonic analysis ideas motivated by Littlewood–Paley theory and classical g -functions into abstract operator theory and combined them with spectral theoretic techniques such as bounded imaginary powers and semigroup theory. For the extension to Banach spaces, in particular L^p -spaces, the central paper is by [Cowling, Doust, McIntosh, and Yagi \[1996\]](#).

A full treatment of the H^∞ -calculus could easily cover a volume in itself. In order to keep the presentation within reasonable bounds we chose to present in this volume the construction of the H^∞ -calculus and its characterisation through square function estimates, along with a discussion of those examples that can be treated in a self-contained way with tools from functional analysis and harmonic analysis presently at our disposal. Further topics related to the H^∞ -calculus, such as fractional powers, bounded imaginary powers, sums of operators, and perturbation theory, as well as their applications in the theory of deterministic and stochastic evolution equations, will be treated in the next volume.

Section 10.1

The material of this section is classical and can be found in, e.g., [Denk, Hieber, and Prüss \[2003\]](#), [Kato \[1995\]](#), [Komatsu \[1966\]](#), [Kunstmann and Weis \[2004\]](#). The notion of sectoriality goes back to the work of Kato (in connection with form methods), Balakrishnan, Komatsu and others in the 1960s.

Section 10.2

The Dunford functional calculus for bounded operators is classical and can be found in many functional analysis textbooks. Surveys of the theory of H^∞ -calculus include the monograph [Haase \[2006\]](#), the lecture notes [Denk, Hieber, and Prüss \[2003\]](#), [Kunstmann and Weis \[2004\]](#), and the survey papers [Weis \[2006\]](#) and [Batty \[2009\]](#).

The uniform boundedness in Calderón reproducing formula, Proposition [10.2.5](#), is essentially from [McIntosh \[1986\]](#); our proof uses an idea of [Haase \[2006, Proposition 5.2.4\]](#). The version for functions f with $f(z)\log(z)$ in $H^1(\Sigma)$ may be new.

Early examples of sectorial operators without a bounded H^∞ -calculus were given in [McIntosh and Yagi \[1990\]](#). The idea of using conditional bases for this

goes back to Baillon and Clément [1991], Venni [1993], Lancien [1998]. For the existence of such bases in every Banach space with a Schauder basis see Pelczyński and Singer [1964/1965] and the exposition in Albiac and Kalton [2006, Section 9.5].

The connection between UMD and the bounded H^∞ -calculus of the Laplacian Δ on $L^p(\mathbb{R}^d; X)$ and the bisectorial operator $-id/dx$ on $L^p(\mathbb{R}; X)$ was pointed out in Hieber and Prüss [1998].

Theorem 10.2.24 is stated in between the lines in McIntosh [1986], where also the contributions Yagi [1984] are credited. The direct proof presented here is due to E. Franks and appears in Albrecht, Duong, and McIntosh [1996]. An extension of the theorem to the Banach space setting was obtained in Kriegler and Weis [2010], where a characterisation of the contractivity of the H^∞ -calculus, i.e., the validity of the inequality $\|f(A)\| \leq \|f\|_\infty$, is given in terms of Blaschke products.

Section 10.3

The notion of R -sectoriality was introduced in Clément and Prüss [2001] and Weis [2001a]. The characterisation of R -sectoriality of Proposition 10.3.3 is due to Weis [2001b]. Theorem 10.3.4 was proved in Kalton and Weis [2001]. In particular this paper contains the result that the boundedness of the H^∞ -calculus implies R -sectoriality for spaces with the triangular contraction property. Another result in this direction, due to Clément and Prüss [2001], states that a sectorial operator with bounded imaginary powers on a UMD space is R -sectorial.

The connection between unconditional decompositions and H^∞ -calculi explained in Lemma 10.3.8 is one of the main themes in Kalton and Weis [2001]. Lemma 10.3.13 appears in Franks [1997]. It extends to the operator-valued H^∞ -calculus (an extension of the H^∞ -calculus to be treated in the next volume) and in this form it appears in Kalton and Weis [2001] and Kalton and Weis [2016]. Corollary 10.3.15 appears in Franks [1997]. Extensions and related results can be found in Boyadzhiev and deLaubenfels [1992], Cowling, Doust, McIntosh, and Yagi [1996], Kalton and Weis [2001].

Permanence of R -sectoriality under real and complex interpolation was proved in Kaip and Saal [2012]. By using Proposition 8.4.4 one can show that consistent R -sectorial operators A_0 and A_1 on an interpolation couple of K -convex Banach spaces interpolate, under both the complex and the real method, to R -sectorial operators with an interpolated angle of R -sectoriality $\omega_R(A_\theta) = (1 - \theta)\omega_R(A_0) + \theta\omega_R(A_1)$.

Section 10.4

We refer the reader to Stein [1970b] for an in-depth discussion of the classical square functions arising in harmonic analysis. McIntosh [1986] and Yagi [1984]

introduced them as a tool in operator theory and considered square functions of the form

$$\|x\|_{\phi,A}^2 = \int_0^\infty \|\phi(tA)x\|^2 \frac{dt}{t}$$

for $\phi \in H_0^\infty(\Sigma)$, the space of bounded holomorphic functions on Σ having polynomial decay near zero and at infinity. This work was subsequently generalised to Banach spaces, first by Cowling, Doust, McIntosh, and Yagi [1996] and subsequently Le Merdy [2004], Kalton and Weis [2016]. See also the survey Le Merdy [2007].

Most of the results of Section 10.4.a are due to Kalton and Weis [2001], Kunstmann and Weis [2004] and Kalton, Kunstmann, and Weis [2006]. Proposition 10.4.6 is the discrete analogue of Proposition 10.4.17, which was proved by Le Merdy [2003, Theorem 1.1] in L^p -spaces and at the same time for Banach spaces by Kalton and Weis [2016, Proposition 7.7] in the form stated here.

The continuous square function results for general Banach spaces of Section 10.4.b are due to Kalton and Weis [2016]. For Hilbert spaces they imply Theorem 10.4.21, which is due to McIntosh [1986] and for L^p -spaces they imply Theorem 10.4.23. For formulation of Theorem 10.4.23 as presented here is from Le Merdy [2004]; a version without any R -sectoriality condition on A is in Cowling, Doust, McIntosh, and Yagi [1996], but there only certain specific functions ϕ are allowed in the implication (2) $\Rightarrow$ (1) (square function estimates imply H^∞ -calculus).

Lemma 10.4.18 is a variation of Kalton, Kunstmann, and Weis [2006, Lemma 4.7]. Proposition 10.4.15, on the equivalences between discrete and continuous square functions, provides an effective way to go back and forth between the discrete and the more difficult continuous time case. The domination of dual square function by discrete square functions of Proposition 10.4.15(3) seems to be new.

For general Banach spaces, strict inequality $\omega(A) < \omega_{H^\infty}(A)$ may hold. A first example without dense range was given in Cowling, Doust, McIntosh, and Yagi [1996]. Subsequently in Kalton [2003] an example was constructed on a uniformly convex space, and in Kalton and Weis [preprint] on a closed subspace of L^p with $p \in (1, 2)$. It is still an open problem whether such an example can exist on an L^p -space with $1 < p < \infty$, $p \neq 2$.

The equality $\omega_{H^\infty}(A) = \omega_R(A)$ for Banach spaces with the triangular contraction property, due to Kalton and Weis [2001], is a reasonable extension of the Hilbert space result $\omega_{H^\infty}(A) = \omega(A)$. Theorem 10.4.22 is due to Le Merdy [1996, 1998] and (for invertible A) Grabowski and Callier [1996].

To avoid assumptions on the Banach space X , Kalton, Kunstmann, and Weis [2006] coined the notion of *almost R -sectoriality*. We briefly indicate how one can weaken the assumptions in some of the results of Section 10.4 by using this notion. By definition, a sectorial operator A has this property if the set

$$\{\lambda AR(\lambda, A)^2 : |\arg \lambda| > \omega\}$$

is R -bounded for some $\omega \in (\omega(A), \pi)$. R -sectoriality implies almost R -sectoriality, but the converse is false by [Kalton and Weis \[preprint\]](#). Let us denote the abscissa of almost R -sectoriality by $\tilde{\omega}_R(A)$. With the help of Theorem [8.5.4](#) it is shown that every sectorial operator with a bounded H^∞ -calculus on a Banach space X is almost R -sectorial, and the equality $\omega_{H^\infty}(A) = \tilde{\omega}_R(A)$ holds. More generally, every sectorial operator with bounded imaginary powers is almost R -sectorial (but not necessarily R -sectorial). The following useful property of almost R -sectorial operators (which is more general than Theorem [10.3.4\(1\)](#)) is also shown: the set $\{\varphi(tA) : t > 0\}$ is R -bounded for all $\varphi \in H^1(\Sigma_\sigma)$ with $\tilde{\omega}_R(A) < \sigma < \pi$.

Theorem [10.4.25](#) is essentially due to [Le Merdy \[2014\]](#), who treated only the case of finite-dimensional Hilbert spaces and formulated the result in a somewhat different but equivalent form. The present formulation is from [Haak and Haase \[2013\]](#), which systematically develops the ℓ^1 -summability provided the Franks–McIntosh Theorem [H.3.1](#) used in the proof.

Theorem [10.4.28](#) is due to [Le Merdy \[2012\]](#). The use of the specific square function corresponding to $\phi(z) = z^{1/2} \exp(-z)$ leads to a characterisation of the H^∞ -calculus with a sharper angle than is obtained by the characterisation in [Cowling, Doust, McIntosh, and Yagi \[1996\]](#). Note that no R -sectoriality assumption is made in the statement of Theorem [10.4.28](#). In [Le Merdy \[2012\]](#), the result is presented only in an L^p -setting, but the possibility to extend it to Banach spaces with Pisier’s contraction property was already pointed out in the paper. For sectorial operators on a Hilbert spaces, the theorem reduces to a result of [McIntosh \[1986\]](#). A discrete version of the theorem can be found in [Le Merdy \[2014\]](#).

Ritt operators

We have treated both continuous square functions and their discrete analogues, and in fact we used the former as a basis for discussing the latter. In a different direction, it is possible to ‘discretise’ the H^∞ -calculus on the operator level. A *Ritt operator* is a bounded operator T acting on a Banach space X which is *power bounded*, i.e., $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$, and satisfies

$$\sup_{n \geq 1} n \|T^n - T^{n-1}\| < \infty.$$

Ritt operators take the role of sectorial operators for the discrete-time problem

$$\begin{cases} u_{n+1} &= Tu_n + f_n, & n \in \mathbb{Z}, \\ u_0 &= 0. \end{cases}$$

Early studies include [Blunck \[2001\]](#), [Kalton and Portal \[2008\]](#), [Portal \[2003\]](#). Subsequently, the H^∞ -calculus for Ritt operators has been developed by [Kalton and Portal \[2008\]](#), [Le Merdy \[2014\]](#), [Lancien and Le Merdy \[2015\]](#), [Arhancet and Le Merdy \[2014\]](#), [Arhancet, Fackler, and Le Merdy \[2017\]](#). It

was shown in [Le Merdy \[2014\]](#) a Ritt operator T on a space L^p with $1 < p < \infty$ admits a bounded H^∞ -calculus if and only if the square functions estimate

$$\left\| \left(\sum_{n \geq 1} n \|T^n f - T^{n-1} f\|^2 \right)^{1/2} \right\|_{L^p} \approx \|f\|_{L^p}$$

holds. This result was extended to Ritt operators on general Banach spaces by [Schwenninger \[2016b\]](#). In such results, the role of sectors is taken by Stolz domains. By definition, the *Stolz domain* of angle $\omega \in (0, \frac{1}{2}\pi)$ is the interior $\mathbb{B}_\omega$ of the convex hull of the point 1 and the disc $B(0, \sin \omega)$. A bounded operator $T \in \mathcal{L}(X)$ is a Ritt operator if and only if there exists an $\omega \in (0, \frac{1}{2}\pi)$ such that $\sigma(T) \subset \overline{\mathbb{B}_\omega}$ and

$$\sup_{\lambda \in \mathbb{C} \setminus \overline{B_\omega}} \|(\lambda - 1)R(\lambda, T)\| < \infty,$$

and in this case $I - T$ is ω -sectorial; see [Lyubich \[1999\]](#), [Nagy and Zemánek \[1999\]](#) and the discussion in [Le Merdy \[2014\]](#).

Conical square functions

Besides the square functions functions that we have considered in this treatment, square function norms of the form

$$\left(\int_{\mathbb{R}^n} \left(\iint_{|y-x| < t} |F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p}, \quad (10.57)$$

where frequently $F(y, t) = (\psi(tA)f)(y)$, play an important role in the theory of H^∞ -calculus. The object in (10.57) is often called a *conical square function* (norm), due to the cone-like shape of the integration domain $\{(y, t) \in \mathbb{R}^n \times (0, \infty) : |y - x| < t\}$. The square functions of the type we have considered, where the quadratic integral extends over $t \in (0, \infty)$ only, are sometimes called *vertical square functions*, in order to make the distinction. In classical harmonic analysis, one can also find several “non-tangential” square functions, like the g_λ^* function of [Stein \[1970a, Section IV.2\]](#), where the quadratic integral extends over all $(y, t) \in \mathbb{R}^d \times (0, \infty)$ but involves a decaying factor as (y, t) gets further away from the cone, but this direction would take us too far afield.

In the context of functional calculus, conical square functions as in (10.57) are particularly useful when dealing with operators without reasonable kernel bounds, such as the elliptic operators of the form $-\operatorname{div} B \nabla$ studied in the Kato square root problem (see also the Notes to Section 10.6). In general such operators are not sectorial on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$ (see [Auscher \[2007\]](#) for more on this). A sample of the rapidly growing literature on this subject is [Auscher, Kriegler, Monniaux, and Portal \[2012b\]](#), [Auscher, McIntosh, and Russ \[2007\]](#), [Auscher, Monniaux, and Portal \[2012c, 2015\]](#), [Hofmann and Mayboroda \[2009\]](#), [Frey, McIntosh, and Portal \[2014\]](#). A comparison of square

functions with the classical ‘vertical’ square functions is made in [Auscher, Hofmann, and Martell \[2012a\]](#).

Conical square functions are closely related to the class of so-called *tent spaces* $T^{p,q}$ introduced by [Coifman, Meyer, and Stein \[1985\]](#). The space $T^{p,2}$ with $q = 2$ has the closest relation to square functions. [Harboure, Torrea, and Viviani \[1991\]](#) showed that these spaces can be isomorphically embedded as complemented subspaces of a suitable L^p -space of L^q -valued functions. This fact was used in [Hytönen, Van Neerven, and Portal \[2008b\]](#) to introduce the vector-valued tent space $T^{p,2}(X)$ for UMD spaces X and to show that certain singular integral operators are boundedness from $L^p(X)$ to $T^{p,2}(X)$.

Section 10.5

Theorem 10.5.1 is due to [Guerre-Delabrière \[1991\]](#); it is a variant of a result of [Bourgain \[1983\]](#), treated earlier as Theorem 5.2.10, which derives the UMD property from the boundedness of the Hilbert transform instead of that of $(-\Delta)^{is}$. The quantitative bound given in Theorem 10.5.1 is not stated by [Guerre-Delabrière \[1991\]](#), but it is implicit in her argument; this was observed in [Hytönen \[2007b\]](#).

Proposition 10.5.4 is due to [Hytönen and Weis \[2008\]](#), but the present quantitative statement ‘with constant 1’ is new. The ergodic Lemma 10.5.6, which is a standard result from ergodic theory adapted from [Katok and Hasselblatt \[1995\]](#), replaces in the present treatment an ad hoc argument in [Hytönen and Weis \[2008\]](#) to approximate arbitrary coefficients $\alpha_{jk} = \pm 1$ as in the real version of Pisier’s contraction property.

Section 10.6

Among the many works on bisectorial operators we mention [Albrecht, Duong, and McIntosh \[1996\]](#), [Auscher, McIntosh, and Nahmod \[1997\]](#), [Arendt and Bu \[2005\]](#), [Dore and Venni \[2005\]](#), [Duelli and Weis \[2005\]](#), [Axelsson, Keith, and McIntosh \[2006\]](#), [Arendt and Duelli \[2006\]](#), [Arendt and Zamboni \[2010\]](#), [Egert \[2015\]](#).

Alongside with id/dx , important examples of a bisectorial operators are the Hodge–Dirac operator D of Example 10.6.5 and its perturbation AD , given by

$$D := \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix}, \quad AD = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} D = \begin{pmatrix} 0 & -\operatorname{div} \\ B\nabla & 0 \end{pmatrix},$$

where the operators act in $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d; \mathbb{R}^d)$, and B is (the pointwise multiplication by) a matrix-valued function with suitable ellipticity assumptions. These operators were used in the seminal paper of [Axelsson, Keith, and McIntosh \[2006\]](#) to give a new proof and several extensions of the Kato square root conjecture,

$$\|\sqrt{L}u\|_{L^2(\mathbb{R}^d)} \approx \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}, \quad L := -\operatorname{div} B\nabla,$$

originally solved by [Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian \[2002\]](#). The estimate above is formally the boundedness, and boundedness from below, of $\nabla L^{-1/2}$, which can be seen as a component of the matrix $\operatorname{sgn}(AD)$ for the holomorphic function $\operatorname{sgn}(z) := z/\sqrt{z^2}$.

Related work on Hodge–Dirac operators include applications to boundary value problems by [Auscher, Axelsson, and Hofmann \[2008\]](#), [Amenta and Auscher \[2016\]](#), [Auscher and Stahlhut \[2016\]](#), the Stokes operator by [McIntosh and Monniaux \[2016\]](#), non-symmetric Ornstein–Uhlenbeck operators by [Maas and Van Neerven \[2009\]](#), and spin manifolds by [Bandara, McIntosh, and Rosén \[2016\]](#), [Bandara and Rosén \[2017\]](#).

The new proof of the Kato conjecture by [Axelsson, Keith, and McIntosh \[2006\]](#) using first-order methods was extended to the setting of $L^p(\mathbb{R}^d; X)$ by [Hytönen, McIntosh, and Portal \[2008a\]](#); here X is required to be a UMD space and in addition to satisfy the RMF property, discussed in Section 3.6.b, which was in fact first invented for the purposes of this paper. Besides the X -valued extension, this paper offered an alternative approach to the L^p theory of the Kato conjecture, which has been further explored by [Auscher and Stahlhut \[2013\]](#), [Frey, McIntosh, and Portal \[2014\]](#), [Hytönen and McIntosh \[2010\]](#), and [Hytönen, McIntosh, and Portal \[2011\]](#).

Section 10.7

Both Transference Theorems 10.7.5 and 10.7.9 are due to [Coifman and Weiss \[1976\]](#). Important contributions to vector-valued extensions of Theorem 10.7.5 include [Berkson, Gillespie, and Muhly \[1986, 1989\]](#), [Asmar, Berkson, and Gillespie \[1990\]](#). Theorem 10.7.10 is due to [Hieber and Prüss \[1998\]](#). Our approach via the Phillips calculus is inspired by [Haase \[2006, Chapter 3\]](#), where the formula (10.45) appears. Extensions of Theorems 10.7.9 and 10.7.10 to unbounded C_0 -groups are obtained in [Haase \[2007, 2009\]](#), and to general C_0 -semigroups in [Haase \[2011\]](#). Among others, in the latter paper it is shown that if $(S(t))_{t \geq 0}$ is any C_0 -semigroup on a Banach space X and if $g \in L^1(\mathbb{R}_+)$ has support in $[a, b] \subseteq (0, \infty)$, then

$$\left\| \int_0^\infty g(t) S(t)x \, dt \right\| \lesssim_p (1 + \log(b/a)) \left(\sup_{t \geq 0} \|S(t)\| \right)^2 \|K_g\|_{\mathcal{L}(L^p(\mathbb{R}_+; X))},$$

where K_g denotes convolution with g . This result is subsequently used to deduce a continuous-time version of results of [Peller \[1982\]](#) and [Vitse \[2005\]](#) providing bounds for the Phillips calculus $\Psi_A(f)$ for the generators $-A$ of polynomially bounded C_0 -semigroups in terms of suitable Besov norms of f . Further extensions and consequences of [Haase \[2011\]](#) can be found in [Haase and Rozendaal \[2013\]](#), [Schwenninger \[2016a\]](#). An extension of the Phillips functional calculus using techniques from systems theory has been obtained in

Zwart [2012] in a Hilbert space setting. It has been extended to the Banach space setting by Schwenninger and Zwart [2012].

Our proof of the second Transference Theorem 10.7.9, on positive contractions on L^p , essentially follows the original one from Coifman and Weiss [1976]. This has the advantage, over another variant explained shortly, that one only needs the Akcoglu–Sucheston Dilation Theorem I.1.2 for the finite-dimensional spaces ℓ_n^p , and not the general version for L^p -spaces (which is obtained from the ℓ_n^p version through ultrapower techniques). A shortcut to Theorem 10.7.9, taken in Kunstmann and Weis [2004], is to use the Fendler’s semigroup version of the Akcoglu–Sucheston theorem:

Theorem 10.8.1 (Fendler). *Every C_0 -semigroup of positive contractions on a space $L^p(S, \mu)$ with $1 < p < \infty$ has a dilation to a C_0 -group of positive isometries on $L^p(T, \nu)$.*

To be more precise, Fendler’s theorem asserts the existence of a measure space $(T, \mathcal{B}, \nu)$, a C_0 -group of positive isometries $(U(t))_{t \in \mathbb{R}}$ on $L^p(T, \nu)$, a positive isometric embedding $J : L^p(S, \mu) \rightarrow L^p(T, \nu)$, and a positive norm one projection P of $L^p(T, \nu)$ onto the range of J such that

$$JS(t) = PU(t)J, \quad t \geq 0.$$

The only known proof of this theorem heavily relies on ultrapower techniques.

Boundedness of the imaginary powers of the negative generator $-A \otimes I_X$ and an extension of Proposition 10.7.8 to the UMD-valued setting were obtained in Clément and Prüss [1990]. In Clément and Prüss [2001], Corollary 10.7.11 (with a different proof) was also noted.

Theorem 10.7.13 is from Kalton and Weis [2001] and was foreshadowed in Weis [2001a]; we follow the arguments in Kunstmann and Weis [2004]. This theorem has a long history. Using advanced methods from harmonic analysis, in particular square function estimates, Stein [1970b] showed (implicitly in section IV.6) that if $-A$ is the generator of a symmetric diffusion semigroup (a precise definition is given in the discussion of Problem P.4) on $L^2(S)$, where $(S, \mathcal{A}, \mu)$ is a σ -finite measure space, then A has a bounded H^∞ -calculus on $L^p(S)$ with $\omega_{H^\infty}(A) < \frac{1}{2}\pi$ for all $1 < p < \infty$. Cowling [1983] gave a more direct proof of this fact, using the Coifman–Weiss transference, and eliminated assumption $T(t)\mathbf{1} = \mathbf{1}$. Carbonaro and Dragičević [2017] obtained the bounded H^∞ -calculus of angle $\omega_{H^\infty}(A) \leq \phi_p^* := \arcsin(1 - \frac{2}{p})$, which is optimal if $S(t)$ is the Ornstein–Uhlenbeck semigroup (see García-Cuerva, Mauceri, Meda, Sjögren, and Torrea [2001, Theorem 2] and Hebisch, Mauceri, and Meda [2004, Theorem 2.2]).

In another direction, Duong and Yan [2002] and Hieber and Prüss [1998] made a big step forward by showing that A has a bounded H^∞ -calculus with $\omega_{H^\infty}(A) \leq \frac{1}{2}\pi$ if $-A$ generates a positive contractive semigroup $S(t)$ on a general L^p -space for just one $p \in (1, \infty)$. Finally it was shown in Kalton and Weis [2001], Weis [2001a] (see also Kunstmann and Weis [2004]) that,

if in addition $S(t)$ extends to an analytic C_0 -semigroup bounded on a sector Σ_δ , then $\omega_{H^\infty}(A) < \frac{1}{2}\pi$. Vector-valued extensions to special classes of UMD-spaces were obtained by Xu [2015]. The H^∞ -calculus for ‘diffusions’ on non-commutative L^p -spaces has been studied Junge, Le Merdy, and Xu [2003].

Lemma 10.7.17 is a variant of the complex interpolation of Stein [1956]; see Voigt [1992] for an abstract version.

Several dilation results have been obtained in Arhancet, Fackler, and Le Merdy [2017]. They showed that if A admits a bounded $H^\infty(\Sigma_\sigma)$ -calculus on some L^p -space, with $1 < p < \infty$, for some $0 < \sigma < \frac{1}{2}\pi$, then $(S(t))_{t \geq 0}$ dilates to a bounded analytic semigroup $(R(t))_{t \geq 0}$ a larger L^p -space in such a way that $R(t)$ is a positive contraction for all $t \geq 0$. Moreover, on reflexive Banach spaces X such that both X and X^* have finite cotype, they show that there is a dilation to a group, which has additional structure if the semigroup generated by $-A$ is positive, or if $X = L^p$. These results elaborate previous results of Fröhlich and Weis [2006].

Further examples of operators with an H^∞ -calculus

Here we collect some further examples of operators with a bounded H^∞ -calculus.

Elliptic partial differential operators

Consider an elliptic system of the form

$$(Ax)(u) = \sum_{|\alpha| \leq 2m} a_\alpha(D^\alpha x)(u), \quad u \in D,$$

where D is an open domain in $\mathbb{R}^d$ satisfying appropriate regularity conditions. For $D = \mathbb{R}^d$ and Hölder continuous coefficients, proofs of the boundedness of the H^∞ -calculus of A on $L^p(D, \mathbb{C}^n)$, $1 < p < \infty$, were given in Amann, Hieber, and Simonett [1994], Denk, Hieber, and Prüss [2003], Kalton, Kunstmann, and Weis [2006], for continuous coefficients in Duong and Simonett [1997], and for VMO-coefficients in Duong and Yan [2002], Heck and Hieber [2003]. Boundedness of the H^∞ -calculus of elliptic operators satisfying suitable kernel bounds on rather general classes of metric measure spaces was shown in Duong and Robinson [1996], Duong and McIntosh [1999]. In Denk, Dore, Hieber, Prüss, and Venni [2004] it was shown that the L^p -realisation of a sectorial operator governing an UMD-valued elliptic boundary value problem with Lopatinskij–Shapiro boundary conditions admits a bounded H^∞ -calculus on $L^p(D; X)$ for all $1 < p < \infty$, provided the top-order coefficients of A are Hölder continuous and the domain G in $\mathbb{R}^d$ has compact C^{2m} -boundary. The proof is based on kernel estimates and a perturbation result for operators admitting bounded H^∞ -calculus. Nau and Saal [2012] obtained improvements of some of these results in the special setting of cylindrical domains of the

form $\Omega = V_1 \times \dots \times V_n \subseteq \mathbb{R}^d$, where each $V_i \subseteq \mathbb{R}^{d_i}$ is a regular domain and $d_1 + \dots + d_n = d$.

In a different direction, the work of [Duong and Robinson \[1996\]](#), [Duong and McIntosh \[1999\]](#) was generalised in [Blunck and Kunstmann \[2003\]](#), where it was shown that kernel estimates can be replaced by *weighted norm estimates* (also referred to as *off-diagonal bounds* or *Davies–Gaffney estimates*). For fixed $1 \leq p < 2 < q \leq \infty$ assume that

$$\|\mathbf{1}_{B(u, t^{1/m})} S(t) \mathbf{1}_{B(v, t^{1/m})}\|_{L^p \rightarrow L^q} \leq \frac{1}{|B(x, t^{1/m})|^{1/p-1/q}} g\left(\frac{d(u, v)}{t^{1/m}}\right) \quad (10.58)$$

for all $t > 0$ and $u, v \in D$, where $g : [0, \infty) \rightarrow [0, \infty)$ is a non-increasing function that decays faster than any polynomial and $(S(t))_{t \geq 0}$ is the bounded analytic C_0 -semigroup generated by $-A$ on $L^2(D)$. Then (10.58) implies that A has a bounded H^∞ -calculus on $L^s(D)$ for all $s \in (p, q)$. Moreover, denoting by A_s the realisation of A on $L^s(D)$, one has $\omega_{H^\infty}(A_s) = \omega_{H^\infty}(A) = \omega(A)$. This allows the treatment of various classes of examples that cannot be treated via kernel estimate, such as Schrödinger operator with a singular potential, elliptic higher order operators with bounded measurable coefficients, and elliptic second order operator with singular lower order terms. This techniques is very flexible and has proved to be of use in a wide range of applications.

Stokes operators

For a bounded domain $\Omega \subseteq \mathbb{R}^d$ with a smooth boundary $\partial\Omega$ we denote by P_p the Helmholtz projection of $L^p(\Omega)^d$ onto the Helmholtz space $L^{p,\sigma}(\Omega)^d$ of divergence-free vector fields, $1 < p < \infty$. The *Stokes operator* then is defined by $A_p = P_p \Delta$ with Dirichlet boundary values. The boundedness of the H^∞ -calculus of A_p is shown in [Noll and Saal \[2003\]](#) and [Kalton, Kunstmann, and Weis \[2006\]](#) and, based on maximal regularity results, in [Fröhlich \[2001\]](#). For earlier results on bounded imaginary powers for Stokes operators see [Giga \[1985\]](#). Since then the boundedness of the H^∞ -calculus of the Stokes operator and its variants have also been considered on certain unbounded domains with various boundary conditions, see e.g. [Abels \[2005a,b\]](#), [Abels and Terasawa \[2009\]](#), [Farwig and Myong-Hwan \[2007\]](#), [Geißert and Kunstmann \[2015\]](#), [Giga, Gries, Hieber, Hussein, and Kashiwabara \[2017\]](#), [Kunstmann \[2008\]](#), [Maier and Saal \[2014\]](#), [Prüss and Simonett \[2016\]](#). For Lipschitz domains, the Stokes operator only operates - and has a bounded H^∞ -calculus, on a part of the L^p -scale, cf. [Kunstmann and Weis \[2017\]](#), [McIntosh and Monniaux \[2016\]](#).

L^1 and $C(K)$ -spaces

In $L^1(\mu)$ and $C(K)$ there are no interesting examples of unbounded operators with a bounded H^∞ -calculus, and certainly no differential operators. See [Kalton and Weis \[2001\]](#), [Hoffmann, Kalton, and Kucherenko \[2004\]](#) and [Kucherenko and Weis \[2005\]](#) for some results in this direction. This situation reflects the well known fact that singular integral operators are generally unbounded on $L^1(\mathbb{R}^d)$ -spaces and $C_b(\mathbb{R}^d)$ -spaces.

Besov spaces and Triebel–Lizorkin spaces

In [Dore \[1999, 2001\]](#) it is shown that if A is a densely defined sectorial operator with dense range on a Banach space X , then A induces sectorial operators on the real interpolation spaces $(X, D(A))_{\theta, q}$ which have a bounded H^∞ -functional calculus. If $X = L^p(\mathbb{R}^d)$ and $D(A) = W^{2,p}(\mathbb{R}^d)$, then $(X, D(A))_{\theta, q}$ equals the Besov spaces $B_{p, q}^s(\mathbb{R}^d)$ with $1 \leq p, q \leq \infty$ (see [Bergh and Lfstrm \[1976\]](#)). [Haase and Rozendaal \[2016\]](#) obtain a functional calculus for generators of bounded and unbounded groups on real interpolation spaces using functions from the analytic Mihlin algebra. In [Kalton and Kucherenko \[2010\]](#) the so-called *absolute functional calculus* was introduced for a sectorial operator A . This calculus has much stronger properties than the H^∞ -calculus. The authors show that a sectorial operator has an absolute functional calculus if and only if the underlying space is a certain real interpolation space.

[Kunstmann, P. C. and Ullmann \[2014\]](#) introduce generalised Triebel–Lizorkin spaces associated with sectorial operators in Banach function spaces and show that an ℓ^s -sectorial operator has a bounded H^∞ -functional calculus in its associated generalised Triebel–Lizorkin spaces.

The boundedness of H^∞ -calculus for pseudodifferential operators in various function spaces on manifolds with or without corners is studied in [Bilyj, Schrohe, and Seiler \[2010\]](#), [Escher and Seiler \[2008\]](#), [Coriasco, Schrohe, and Seiler \[2007\]](#), [Denk, Saal, and Seiler \[2009\]](#).