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Estimating tails of independently stopped random walks using concave approximations of hazard functions

Jaakko Lehtomaa

Abstract

This paper considers logarithmic asymptotics of tails of randomly stopped sums. The stopping is assumed to be independent of the underlying random walk. First, finiteness of ordinary moments is revisited. Then, the study is expanded to more general asymptotic analysis. Results are applicable to a large class of heavy-tailed random variables. The main result enables one to identify if the asymptotic behaviour of a stopped sum is dominated by its increments or the stopping variable. As a consequence, new sufficient conditions for the moment determinacy of compounded sums are obtained.

Keywords: Heavy-tailed; Logarithmic asymptotics; Moment determinacy; Random stopping; Stieltjes moment problem

AMS subject classification: 60G50; 60E05; 60F10; 44A60

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where all the subsequent random variables are defined. Suppose N is a random variable on the set $\mathbb{N} = \{1, 2, 3, \dots\}$. The variable

$$S_N = \sum_{k=1}^N X_k \tag{1.1}$$

is called a *randomly stopped sum*. Here $(X_i) := (X_i)_{i=1}^\infty$ is a sequence of real valued random variables called *increments*. If (X_i) is an i.i.d. sequence, then the variable of Formula (1.1) is called a *stopped random walk*.

Randomly stopped sums are some of the most studied randomly stopped processes [10, 11]. Typically, one is interested to know how the tail of S_N is affected by its increments and the stopping variable. Identification of the dominant variable of the stopped sum has applications in e.g. insurance, where the compounded variable S_N is used to model the aggregate loss of a company. In this setting, the aim is to find out if large losses are mainly caused by few big claims or unusually large amounts of small claims.

Asymptotic form of the tail of S_N has been studied extensively under different distributional assumptions. Specifically, the central topic has been to study if the tail behaviour of $\mathbb{P}(S_N > x)$, as $x \rightarrow \infty$, corresponds to that of

I $\mathbb{E}(N)\mathbb{P}(X_1 > x)$ or

II $\mathbb{P}(\mathbb{E}(X_1)N > x)$.

In I, the increment X is dominating while in II the dominating variable is N . The case I has been studied in [3, 5, 7, 20]. The case II seems to be less studied than case I. Although, references for the latter can be found from [4, 19].

The main contribution of the present paper is to offer a simple method that allows one to identify, on a rough scale, which situation I or II is present. The paper provides a partial answer to the question *When does the increment X or the stopping N alone dominate the asymptotic behaviour of the stopped random walk S_N ?* We begin with results about moments and expand the study to results concerning asymptotic scales. This leads to an intuitively satisfying result, in Theorem 2.2, where heavy increments X lead to I while heavy stopping N leads to II.

The second result is an application of the first result and concerns the moment determinacy of random sums. It turns out that the asymptotic scale of the random sum is closely related to the moment determinacy of S_N . A random variable $X \geq 0$ with finite moments $\mathbb{E}(X^k)$ of all orders $k \in \mathbb{N}$ is *determined by its moments* if there is no other law besides the law of X , $\mathcal{L}(X)$, having the same moment sequence. The topic of moment determinacy is widely studied [9, 15, 16, 21, 23].

The emphasis of previous research has been on discovering sufficient or necessary conditions for moment determinacy [1, 18, 22]. The moment determinacy of random sums has been studied in [15]. As a theoretical consequence of the proof technique developed for Theorem 2.2, new sufficient conditions are derived for the moment determinacy of S_N . These conditions can be verified using the tail functions of X and N . In contrast to most of the previous research, the new approach works equally well when N is heavy-tailed or even dominating the behaviour of S_N .

1.1 Structure of the Paper

Preliminary information is recalled and defined in Section 1.2. All of the results are presented in Section 2. For the convenience of the reader, explanations and applications illustrating the results are presented directly after the exposition of the results. Proofs are postponed to Section 4. Technical details omitted during the main text are presented as appendices in Section A.

1.2 Preliminaries and Definitions

In [14], it is established that the heaviness of a random variable is connected to a concave approximating function called *natural scale*. Here, we use the more

descriptive name *lower concave approximation* for this function. This is recalled in Lemma 1.1 below. A lower concave approximation of the random variable X is a concave approximation of the *hazard function*

$$R_X(x) := -\log \mathbb{P}(X > x),$$

where $:=$ denotes equality by definition. The key observation is that the hazard function of any heavy-tailed random variable admits such an approximation.

Definition 1.1. A random variable X is (right) *heavy-tailed*, if $\mathbb{E}(e^{sX}) = \infty$ for all $s > 0$. A random variable that is not heavy-tailed, is *light-tailed*.

The index is omitted if the random variable related to h or R is clear from context. The same holds for a general member of an i.i.d. sequence. Most results are formulated for right tails. To study left tails one can replace X by $-X$. For a real number x , set $x^+ := \max(0, x)$ and $x^- = \max(0, -x)$. Standard asymptotic notations are used.

We recall the following concept from [14].

Definition 1.2 (*Lower concave approximation*). Let be X non-negative and heavy-tailed. Suppose h is a strictly increasing concave function $h_X: [0, \infty) \rightarrow [0, \infty)$ such that $h_X(0) = 0$, $h_X(x) \rightarrow \infty$ and $h_X(x) = o(x)$, as $x \rightarrow \infty$. In particular, suppose that

$$\liminf_{x \rightarrow \infty} \frac{R(x)}{h_X(x)} = 1. \quad (1.2)$$

Such a function h_X is called a *lower concave approximation* of the random variable X . Note that there are many lower concave approximations corresponding to a single random variable.

Parts 1-3 of the following lemma are recalled from Theorem 1 of [14] while Part 4 is proved separately in Section A.1.

Lemma 1.1. *Suppose X is non-negative and heavy-tailed.*

1. *X has a lower concave approximation.*
2. *For any continuous and strictly increasing function $h: [0, \infty) \rightarrow [0, \infty)$ such that $h(x) \rightarrow \infty$, as $x \rightarrow \infty$, it holds that*

$$\mathbb{I}_h(X) := \liminf_{x \rightarrow \infty} \frac{R(x)}{h(x)} = \sup\{s \geq 0 : \mathbb{E}(e^{sh(X)}) < \infty\}. \quad (1.3)$$

3. *For a lower concave approximation h_X it holds that*

$$\begin{cases} \mathbb{E}(e^{sh_X(X)}) < \infty, & \text{if } s < 1 \\ \mathbb{E}(e^{sh_X(X)}) = \infty, & \text{if } s > 1. \end{cases}$$

4. *Suppose concave functions $f_1, f_2: [0, \infty) \rightarrow [0, \infty)$ with $f_1(0) = f_2(0) = 0$ and*

$$\begin{cases} \mathbb{E}(e^{f_1(X)}) = \infty \text{ and} \\ \mathbb{E}(e^{f_2(X)}) < \infty \end{cases}$$

are given. Then, it is possible to choose lower concave approximations h_X and h_X^* so that $f_1 \geq h_X$ and $h_X^* \geq f_2$.

The following lemma lists some of the basic properties of lower concave approximations.

Lemma 1.2. *The following properties are always valid:*

1. *If h is a lower concave approximation of X and $g \sim h$, then*

$$\liminf_{x \rightarrow \infty} R_X(x)/g(x) = 1.$$

2. *If h, r are lower concave approximations of X , then*

$$\liminf_{x \rightarrow \infty} h(x)/r(x) \leq 1.$$

3. *If $c > 0$ and h_X is a lower concave approximation of X then g is a lower concave approximation of cX , where $g(x) = h_X(x/c)$.*

Proof. Clear from definitions and Lemma 1.1. □

2 Results

The following assumptions are used in Section 2:

- I The sequence of increments (X_i) is an i.i.d. sequence
- II The stopping variable N is independent of (X_i) .

Under assumptions I and II the variable S_N of Equation (1.1) is an independently stopped random walk.

2.1 Moments of Randomly Stopped Sums

The *moment index* $\mathbb{I}(X)$ of a random variable X is defined as

$$\mathbb{I}(X) := \sup\{s \geq 0 : \mathbb{E}((X^+)^s) < \infty\} \in [0, \infty]. \quad (2.1)$$

A small moment index signifies high risk in the sense of a heavy (right) tail. In Proposition 2.1, the moment index of S_N is resolved. The proof of Proposition 2.1 is elementary. It is based on subadditivity and classical inequalities. There exist various results that imply parts of (2.2) in particular under the additional assumption that both of the expectations $\mathbb{E}(|X|)$ and $\mathbb{E}(N)$ are finite, see for example [10, 11]. A unified proof requiring minimal assumptions is presented for the convenience of the reader. In addition, Formula (2.3) provides both upper and lower bounds for the case of unbounded expectations.

Proposition 2.1. *Suppose assumptions I and II hold. Assume $S_n \rightarrow \infty$ almost surely.*

If at least one of the expectations $\mathbb{E}(|X|)$ or $\mathbb{E}(N)$ is finite, then

$$\mathbb{I}(S_N) = \min(\mathbb{I}(X), \mathbb{I}(N)). \quad (2.2)$$

If $\mathbb{E}(X^+) = \mathbb{E}(N) = \infty$, it holds that

$$\mathbb{I}(X)\mathbb{I}(N) \leq \mathbb{I}(S_N) \leq \min(\mathbb{I}(X), \mathbb{I}(N)). \quad (2.3)$$

Remark 2.1. Condition $S_n \rightarrow \infty$ is valid when $\mathbb{E}(X) > 0$. If X is not integrable, the situation is more delicate. This is discussed in Section 3.1.

In the general case where no assumption is made about the drift of S_n , Proposition 2.1 still enables the following corollary.

Corollary 2.1.1. *Suppose $\mathbb{E}(|X|) < \infty$ and $\mathbb{I}(X) \leq \mathbb{I}(N)$. Then*

$$\mathbb{I}(S_N) = \mathbb{I}(X). \quad (2.4)$$

The interpretation of Corollary 2.1.1 is immediate: At the level of moments, the stopped random walk can exhibit exotic behaviour only when the stopping variable has heavier tail than the increment.

2.2 Scales of Randomly Stopped Sums

This section generalises ideas of Section 2.1. The necessity of the generalisation becomes apparent when the increment or stopping variable has relatively light but still heavy-tailed distribution. Specifically, this is the case with Weibull and lognormal type variables whose moments of all orders are finite.

Theorem 2.2. *Suppose assumptions I and II are valid. Let X and N be heavy-tailed random variables. Assume $0 < \mathbb{E}(X) = \mu < \infty$.*

1. *Suppose $\mathbb{E}((X^+)^{1+\delta}) < \infty$ for some $\delta > 0$. Let h_X be a lower concave approximation of X such that $h_X(x) \geq (1 + \delta) \log x$ for all large enough x . If*

$$\liminf_{x \rightarrow \infty} \frac{h_N(x/c_1)}{h_X(x)} \in [1, \infty] \quad (2.5)$$

holds for some lower concave approximation h_N of N and some $c_1 > \mu$, then h_X is a lower concave approximation of S_N .

2. *Suppose $\mathbb{E}(N^{1+\delta}) < \infty$ for some $\delta > 0$. Let h_N be a lower concave approximation of N such that $h_N(x) \geq (1 + \delta) \log x$ for all large enough x . If*

$$\liminf_{x \rightarrow \infty} \frac{h_X(x)}{h_N(x/\mu)} \in [1, \infty] \quad (2.6)$$

holds for some lower concave approximation h_X of X , then $h_N(x/\mu)$ is a lower concave approximation of S_N .

Corollary 2.2.1. *Suppose $X \geq 0$ almost surely. Assume that h_X satisfies conditions of Theorem 2.2 Part 1. If it also holds that*

$$\lim_{x \rightarrow \infty} \frac{R_X(x)}{h_X(x)} = 1, \quad (2.7)$$

then (2.5) implies $R_{S_N}(x) \sim h_X(x)$, as $x \rightarrow \infty$.

Similarly, if h_X satisfies conditions of Theorem 2.2 Part 2 and

$$\lim_{x \rightarrow \infty} \frac{R_N(x/\mu)}{h_N(x/\mu)} = 1, \quad (2.8)$$

then (2.6) implies $R_{S_N}(x) \sim h_N(x/\mu)$, as $x \rightarrow \infty$.

Remark 2.2. In Theorem 2.2 both of the variables X and N are assumed to be heavy-tailed. Cases where heavy-tailed increment is put against a light-tailed stopping or vice versa, are immediately dominated by the heavy-tailed variable. See Appendix A.2 for details.

Remark 2.3. In Parts 1 and 2 of Theorem 2.2 the functions h_X and h_N that satisfy $h_X(x) \geq (1+\delta) \log x$ and $h_N(x) \geq (1+\delta) \log x$ for all $x \geq 0$ can be found by Lemma 1.1, Part 4. Once the functions are found, only conditions (2.5) and (2.6) remain to be verified.

Remark 2.4. If R_X is a lower concave approximation of X and $R_N(x/\mathbb{E}(X))$ is a lower concave approximation of $\mathbb{E}(X)N$ for some heavy-tailed variables X and $\mathbb{E}(X)N$, then the hazard functions themselves can be used as scales. In this case,

$$\mathbb{P}(\mathbb{E}(X)N > x) = o(\mathbb{P}(X > x)), \text{ as } x \rightarrow \infty$$

implies (2.5) while

$$\mathbb{P}(X > x) = o(\mathbb{P}(\mathbb{E}(X)N > x)), \text{ as } x \rightarrow \infty$$

implies (2.6) for $h_X(x) = R_X(x)$ and $h_N(x/\mu) = R_N(x/\mu)$.

Remark 2.5. Scales enable more delicate tail behaviour between the tails of X and N than actual asymptotic dominance as in Remark 2.4. In fact, the tail functions of X and N may even cross infinitely many times as long as the associated scales satisfy (2.5) or (2.6). This is illustrated in Example 3.4 of Section 3.

2.3 Moment Determinacy of Random Sums

As recalled in the introduction, moment determinacy (in Stieltjes' sense) of a non-negative unbounded random variable X means that for all non-negative random variables Y :

$$\mathbb{E}(X^k) = \mathbb{E}(Y^k), \forall k \in \mathbb{N} \implies X \stackrel{d}{=} Y. \quad (2.9)$$

If (2.9) fails, the distribution of X is called *moment indeterminate*. In this section we wish to find sufficient conditions ensuring that the compounding operation preserves moment determinacy.

We begin with a lemma that reformulates Hardy's condition presented in [23] via hazard functions. The condition gives an asymptotic test for the moment determinacy involving the concept of lower concave approximation.

Lemma 2.3. *Suppose $X \geq 0$ is a random variable. If*

$$\liminf_{x \rightarrow \infty} \frac{R_X(x)}{\sqrt{x}} \in (0, \infty], \quad (2.10)$$

then X is determined by its moments. In addition, if a lower concave approximation h_X of a heavy-tailed X satisfies

$$\liminf_{x \rightarrow \infty} \frac{h_X(x)}{\sqrt{x}} \in (0, \infty], \quad (2.11)$$

then (2.10) holds and thus X is determined by its moments.

Remark 2.6. In Lemma 2.3 moment determinacy is identified using the asymptotic properties of the tail function making small values of X irrelevant. In addition, no assumption about the absolute continuity w.r.t. the Lebesgue measure is needed. This is different from other tests such as Carleman condition or finiteness of the Krein integral combined with Lin condition [9, 21].

Lemma 2.3 and techniques used in the proof of Theorem 2.2 enable the following result.

Theorem 2.4. *Suppose assumptions I and II are valid. Assume that X and N are heavy-tailed and X is non-negative. Assume further that one of the conditions 1 or 2 holds:*

1. *There exists a lower concave approximation h_X of X such that*

$$\liminf_{x \rightarrow \infty} \frac{h_X(x)}{\sqrt{x}} \in (0, \infty] \quad (2.12)$$

and

$$\liminf_{x \rightarrow \infty} \frac{h_N(x/c_1)}{h_X(x)} \in (0, \infty] \quad (2.13)$$

for some scale h_N of N and some $c_1 > \mathbb{E}(X) = \mu$.

2. *There exists a lower concave approximation h_N of N such that*

$$\liminf_{x \rightarrow \infty} \frac{h_N(x/\mu)}{\sqrt{x}} \in (0, \infty] \quad (2.14)$$

and

$$\liminf_{x \rightarrow \infty} \frac{h_X(x)}{h_N(x/\mu)} \in (0, \infty] \quad (2.15)$$

for some scale h_X of X

Then S_N is determined by its moments.

3 Comments, Examples and Applications

Many of the earlier studies, for example [4, 5, 7], concerning stopped random walks concentrate on situations where the tails of random variables satisfy a no-oscillation condition such as subexponentiality or strong subexponentiality. Under these types of assumptions, the asymptotic approximation of tail probability $\mathbb{P}(S_N > x)$ can often be presented in a sharp form which implies the weaker log-asymptotics of the form, say, $R_{S_N}(x) \sim R_X(x)$ in the case where X is heavier than N . So, checking for (strong) subexponentiality and applying the earlier results gives a natural sufficient condition for the log-asymptotics to hold. If R_X admits a lower concave approximation in the sense of (2.7), the result is consistent with Corollary 2.2.1.

The current study derives a rough asymptotic approximation without assuming any no-oscillation conditions for the tail functions. The task is achieved by considering sufficiently smooth approximations, lower concave approximations, of hazard functions. The results are best suited for cases where the tail functions exhibit oscillations that prevent the application of earlier sharp results or when it is easier to find a concave approximation for the hazard function than to show a suitable no-oscillation condition.

Finding a lower concave approximation can be difficult, but there exist methods to recover such a function. Firstly, the proof of Theorem 1 in [14] is a constructive proof which always produces a function that can be used as the lower concave approximation. However, the constructed function is a piece-wise linear function that does not typically admit a simple analytical expression. The user can attempt to find one as follows. If a hazard function R can be evaluated at any given point, it is sometimes possible to find a concave approximation by visual inspection of the graph of R . In this method, the user forms an initial guess by selecting an increasing sequence of points (x_n) such that the linear interpolation function between the pairs $(x_n, R(x_n))$ has a decreasing growth rate. A concave function with a simple analytic expression can then be fitted to the pairs $(x_n, R(x_n))$. Finally, Condition (1.2) can be checked using this function.

The following section illustrates why some of the assumptions are made and what can go wrong if they are omitted.

3.1 Necessity of the Positive Drift

In Theorems 2.2 and 2.4 a positive drift of the underlying random walk is assumed. If this assumption was changed, the asymptotics of the stopped random walk S_N would be different.

In general, a random walk (S_n) exhibits almost surely one of the following behaviours:

1. $S_n \rightarrow \infty$, as $n \rightarrow \infty$ (positive drift)
2. $S_n \rightarrow -\infty$, as $n \rightarrow \infty$ (negative drift)
3. $\liminf_{n \rightarrow \infty} S_n = -\infty$ and $\limsup_{x \rightarrow \infty} S_n = \infty$.

If the increment X satisfies $\mathbb{E}(|X|) < \infty$, it is known that conditions 1, 2 and 3 are equivalent with conditions $\mathbb{E}(X) > 0$, $\mathbb{E}(X) < 0$ and $\mathbb{E}(X) = 0$, respectively. For this and further characterisations of the conditions 1-3, see [6] or the beginning of [12].

The following example demonstrates what can happen at the level of moments if the increment has a negative expectation.

Example 3.1. Suppose X_1 is regularly varying, $\mathbb{E}(X_1) < 0$ and $\mathbb{E}(N) < \infty$. Then it is well-known that

$$\mathbb{P}(S_N > x) \sim \mathbb{P}(X_1 > x)\mathbb{E}(N), \quad x \rightarrow \infty.$$

So, if $\mathbb{I}(X) > \mathbb{I}(N)$, Equality (2.2) does not hold.

Example 3.2 demonstrates that the behaviour of S_N can also be different from (2.2), when $\mathbb{E}(X) = 0$.

Example 3.2. If $\mathbb{E}(X) = 0$, the Marcinkiewicz-Zygmund inequalities and their generalisations become available. These are used in Section 1.5 of [10]. In fact, in Part (iii) of Theorem 5.1. of [10] the following result is shown: If $\mathbb{E}(|X|) < \infty$ and $\mathbb{E}(X) = 0$ it holds for $s \geq 2$ that

$$\mathbb{E}(|S_N|^s) \leq C_s \mathbb{E}(|X|^s) \mathbb{E}(N^{s/2}), \quad (3.1)$$

where C_s is a positive constant. Formula (3.1) implies

$$\mathbb{I}(S_N) \geq \min(\mathbb{I}(|X|), 2\mathbb{I}(N)) = \min(\mathbb{I}(X^+), \mathbb{I}(X^-), 2\mathbb{I}(N)),$$

when $\min(\mathbb{I}(X^+), \mathbb{I}(X^-)) \geq 2$ and $\mathbb{I}(N) \geq 1$. Clearly, this is not consistent with Equation (2.2).

In contrast to Example 3.2, the next example demonstrates that behaviour consistent with (2.2) is possible even when some of the assumptions are violated. Namely, the stopping N can determine the behaviour of S_N when X is Cauchy distributed.

Example 3.3. Suppose X is Cauchy distributed with density

$$f(x) = \frac{1}{\pi} \frac{1}{(1+x)^2}, \quad x \in \mathbb{R}.$$

Then, it is well known that

$$\frac{S_n}{n} \stackrel{d}{=} X \quad (3.2)$$

holds for any $n \in \mathbb{N}$. Using (3.2) we get

$$\mathbb{E}(((S_N)^+)^s) = \sum_{k=1}^{\infty} k^s \mathbb{E}(((S_k/k)^+)^s) \mathbb{P}(N = k) = \mathbb{E}((X^+)^s) \mathbb{E}(N^s),$$

which implies (2.2).

Examples 3.1, 3.2 and 3.3 together show that the behaviour of moments of S_N is universally determined by (2.2) only when the drift is positive. Without the positive drift, behaviour of S_N can change or stay the same.

Figure 1: Illustration of the construction of functions R_X (blue, large dashes) and R_Y (red, small dashes).

3.2 Zigzagging Tail Functions

As mentioned in Remark 2.5, scales of hazard functions may be used even if the tails of X and N cross infinitely often. This enables application of Theorem 2.2 in a wide range of situations including, but not limited to tail dominance between X and N . In Example 3.4 this type of situation is artificially constructed.

Example 3.4. Suppose we are given functions

$$h_i: [0, \infty) \rightarrow [0, \infty),$$

where $i = 1, 2, 3, 4$ that satisfy

- I $h_i(0) = 0$ and h_i is strictly increasing and continuous for every i
- II $h_1 < h_2 < h_3 < h_4$
- III h_1 and h_2 are concave.

First, we define the sequences (x_n) and (y_n) by requiring

1. $x_0 = y_0 = 1$
2. $x_{n+1} = h_1^{-1}(h_4(x_n))$
3. $y_{n+1} = h_2^{-1}(h_3(x_n))$,

where $n \in \{0, 1, 2, \dots\}$. Then, we construct random variables X and Y , whose probability mass is concentrated to the sets $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$, respectively. We set

$$\mathbb{P}(X = x_n) = e^{-h_4(x_n)} - e^{-h_1(x_n)}$$

and

$$\mathbb{P}(Y = y_n) = e^{-h_3(y_n)} - e^{-h_2(y_n)}$$

for $n \in \{0, 1, 2, \dots\}$. The procedure is illustrated in Figure 1.

Now, by construction, it must hold that

$$\limsup_{x \rightarrow \infty} \frac{R_X(x)}{h_4(x)} = \limsup_{x \rightarrow \infty} \frac{R_Y(x)}{h_3(x)} = \liminf_{x \rightarrow \infty} \frac{R_Y(x)}{h_2(x)} = \liminf_{x \rightarrow \infty} \frac{R_X(x)}{h_1(x)} = 1. \quad (3.3)$$

By continuity of the functions h_i the difference of hazard functions $R_X(x) - R_Y(x)$ must change signs infinitely many times. This implies that the tails must also cross infinitely many times. However, from (3.3) we may still conclude that h_1 is a lower concave approximation of X and h_2 is a lower concave approximation of Y .

4 Proofs

Proof. (Proof of Proposition 2.1) We divide the proof into separate cases.

1. We will first prove (2.2). To see that

$$\mathbb{I}(S_N) \geq \min(\mathbb{I}(N), \mathbb{I}(X)),$$

we can use one of the following tools:

- (a) Subadditivity of the function $x \mapsto x^s$ for $x \geq 0$ and $0 < s \leq 1$,
- (b) Jensen's inequality or
- (c) Minkowski's inequality.

Depending on the value of s , the aim is to bound the expectation $\mathbb{E}((S_N^+)^s)$ from above. Suppose $0 < s \leq 1$. Because of the fact $S_N^+ \leq \sum_{k=1}^N X_k^+$ and the subadditivity it must hold that

$$\begin{aligned} \mathbb{E}((S_N^+)^s) &\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\left(\sum_{m=1}^k X_m^+ \right)^s \right) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} k \mathbb{E}((X^+)^s) \mathbb{P}(N = k) \\ &= \mathbb{E}((X^+)^s) \mathbb{E}(N). \end{aligned} \quad (4.1)$$

On the other hand, applying Jensen's inequality to the concave function $x \mapsto x^s$ we get

$$\begin{aligned}\mathbb{E}((S_N^+)^s) &\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\left(\sum_{m=1}^k X_m^+ \right)^s \right) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} k^s (\mathbb{E}(X^+))^s \mathbb{P}(N = k) \\ &= (\mathbb{E}(X^+))^s \mathbb{E}(N^s).\end{aligned}\tag{4.2}$$

Suppose then $s > 1$. By Minkowski's inequality we have

$$\begin{aligned}\mathbb{E}((S_N^+)^s) &\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\left(\sum_{m=1}^k X_m^+ \right)^s \right) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} k^s \mathbb{E}((X^+)^s) \mathbb{P}(N = k) \\ &= \mathbb{E}((X^+)^s) \mathbb{E}(N^s).\end{aligned}\tag{4.3}$$

Equations (4.1), (4.2) and (4.3) imply $\mathbb{I}(S_N) \geq \min(\mathbb{I}(N), \mathbb{I}(X))$ under the assumption that one of the expectations $\mathbb{E}(X^+)$ or $\mathbb{E}(N)$ is finite.

We will then prove $\mathbb{I}(S_N) \leq \min(\mathbb{I}(N), \mathbb{I}(X))$. Choose m so that $\mathbb{P}(N = m) > 0$. Then

$$(S_N)^+ \geq (X_1 + X_2 + \dots + X_m)^+ \mathbf{1}(N = m),\tag{4.4}$$

and we have that $\mathbb{I}(S_N) \leq \mathbb{I}(X_1 + X_2 + \dots + X_m) = \mathbb{I}(X)$ by the i.i.d assumption and basic calculation rules of moment indices, see e.g. [13]. In the case of a finite expectation we note that for any $a \in (0, \mathbb{E}(X))$ it holds that $\mathbb{P}(S_k > ak) \geq C$ for all k and some $C > 0$ that does not depend on k . This is clear from the law of large numbers. In the remaining case where $\mathbb{E}(|X|) = \infty$ but $S_n \rightarrow \infty$ almost surely we have, see the beginning of e.g. [12], that $S_n/n \rightarrow \infty$ almost surely. Hence, for $a > 0$,

$$\begin{aligned}\mathbb{E}((S_N^+)^s) &= \sum_{k=1}^{\infty} \mathbb{E}((S_k^+)^s) \mathbb{P}(N = k) \\ &\geq \sum_{k=1}^{\infty} \mathbb{E}((S_k^+)^s \mathbf{1}(S_k > ak)) \mathbb{P}(N = k) \\ &\geq a^s \sum_{k=1}^{\infty} k^s \mathbb{P}(S_k > ak) \mathbb{P}(N = k) \\ &\geq a^s C \mathbb{E}(N^s),\end{aligned}$$

which proves $\mathbb{I}(S_N) \leq \mathbb{I}(N)$. This ends the proof of (2.2).

2. We are set to prove (2.3). Since the proof of the inequality $\mathbb{I}(S_N) \leq \min(\mathbb{I}(N), \mathbb{I}(X))$ did not require any assumptions about the finiteness of the expected values $\mathbb{E}(|X|)$ and $\mathbb{E}(N)$, we only need to show $\mathbb{I}(S_N) \geq \mathbb{I}(X)\mathbb{I}(N)$. Without any loss of generality we can assume $\min(\mathbb{I}(N), \mathbb{I}(X)) > 0$.

Recall that $\mathbb{E}(X^+) = \infty$ so that $\mathbb{I}(X) \leq 1$. Fix a number $\epsilon > 0$ so small that for $q_1 := \mathbb{I}(X) - \epsilon$ it holds that $q_1 > 0$. Set $q_2 := \mathbb{I}(N)$ and $q := q_1 q_2$. Now, for $0 < s < 1$, it holds by subadditivity and Jensen's inequality that

$$\begin{aligned}
\mathbb{E}(((S_N^+)^q)^s) &\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\left(\left(\sum_{m=1}^k X_m^+ \right)^{q_1} \right)^{q_2 s} \right) \mathbb{P}(N = k) \\
&\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\left(\sum_{m=1}^k (X_m^+)^{q_1} \right)^{q_2 s} \right) \mathbb{P}(N = k) \\
&\leq \sum_{k=1}^{\infty} k^{q_2 s} (\mathbb{E}((X^+)^{q_1}))^{q_2 s} \mathbb{P}(N = k) \\
&= (\mathbb{E}((X^+)^{q_1}))^{q_2 s} \mathbb{E}(N^{q_2 s}) < \infty.
\end{aligned} \tag{4.5}$$

Inequality (4.5) implies $\mathbb{I}((S_N^+)^q) \geq 1$, which is equivalent to

$$\mathbb{I}(S_N) \geq q = (\mathbb{I}(X) - \epsilon)\mathbb{I}(N).$$

Now, letting $\epsilon \rightarrow 0$ proves the claim. □

Proof. (Proof of Corollary 2.1.1) Set

$$S_N^a := \sum_{k=1}^N (X_k - \mathbb{E}(X_k) + |\mathbb{E}(X_k)| + 1)$$

and

$$S_N^b := \sum_{k=1}^N (X_k - \mathbb{E}(X_k) - (|\mathbb{E}(X_k)| + 1)).$$

Since $S_N^b \leq S_N \leq S_N^a$, it must hold that

$$\mathbb{I}(S_N^b) \geq \mathbb{I}(S_N) \geq \mathbb{I}(S_N^a). \tag{4.6}$$

The underlying random walk of S_N^a is positive, while that of S_N^b is negative.

By Proposition 2.1, $\mathbb{I}(S_N^a) = \min(\mathbb{I}(X - \mathbb{E}(X) + |\mathbb{E}(X)| + 1), \mathbb{I}(N))$. Using the basic properties of moment indices, see e.g. Section 2.2 of [13], and the assumption $\mathbb{I}(N) \geq \mathbb{I}(X)$ we have that $\mathbb{I}(S_N^a) = \mathbb{I}(X)$. In addition, Inequality (4.4) holds in this case as well and we obtain $\mathbb{I}(S_N^b) \leq \mathbb{I}(X - \mathbb{E}(X) - (|\mathbb{E}(X)| + 1)) = \mathbb{I}(X)$. Equation (4.6) proves the claim. □

The proof of Proposition 2.1 was based on application of classical inequalities. Once we move on to the study of general scales, the following two technical lemmas become necessary.

Lemma 4.1. *Suppose that $h: [0, \infty) \rightarrow [0, \infty)$ is an increasing concave function with $h(0) = 0$. Then for all $x \geq 0$:*

$$\begin{cases} h(cx) \geq ch(x), & \text{if } 0 < c < 1 \text{ and} \\ h(cx) \leq ch(x), & \text{if } c > 1. \end{cases}$$

Proof. See Lemma 4 of [14] □

Lemma 4.2. *Let X be a non-negative heavy-tailed random variable and let h_X be a lower concave approximation of X . Then the mapping*

$$c \mapsto \mathbb{I}_{h_X}(cX)$$

is continuous at point $c = 1$.

Proof. By concavity and the property $h_X(0) = 0$, Lemma 4.1 is applicable.

For a fixed $0 < c_1 < 1$ and $s \geq 0$ we may estimate $\mathbb{E}(e^{sh_X(c_1X)}) \geq \mathbb{E}(e^{c_1sh_X(X)})$ and obtain, from e.g. Connection (1.3), that

$$\sup\{s \geq 0 : \mathbb{E}(e^{sh_X(c_1X)}) < \infty\} \leq \sup\{s \geq 0 : \mathbb{E}(e^{sc_1h_X(X)}) < \infty\} = \frac{1}{c_1}. \quad (4.7)$$

Similarly, for a fixed $c_2 > 1$ we have $\mathbb{E}(e^{sh_X(c_2X)}) \leq \mathbb{E}(e^{c_2sh_X(X)})$ and so

$$\sup\{s \geq 0 : \mathbb{E}(e^{sh_X(c_2X)}) < \infty\} \geq \sup\{s \geq 0 : \mathbb{E}(e^{sc_2h_X(X)}) < \infty\} = \frac{1}{c_2}. \quad (4.8)$$

Letting $c_1 \uparrow 1$ in (4.7) and $c_2 \downarrow 1$ in (4.8) we finally get that

$$\lim_{c \rightarrow 1} \left(\sup\{s \geq 0 : \mathbb{E}(e^{sh_X(cX)}) < \infty\} \right) = 1,$$

which proves the claim. □

The following lemma is an extended version of Lemma 3 of [5]. It utilises Lemma 4 of [5] in its proof.

Lemma 4.3. *Suppose $\xi, \xi_1, \xi_2, \dots$ is an i.i.d. sequence of integrable random variables. Let $h: [0, \infty) \rightarrow [0, \infty)$ be an increasing concave function with the property $h(x) \geq \log x$ for all x large enough. Assume further that $h(x) = o(x)$, as $x \rightarrow \infty$ and $\mathbb{E}(e^{h(\xi^+)}) < \infty$.*

Then the following inequalities hold:

1. *If $\mathbb{E}(\xi) = \mu > 0$, then for every $c > \mu$ there exists a constant $K(c)$ depending on the distribution of ξ and the number c , but not on the index n , so that*

$$\mathbb{E}(e^{h((\xi_1 + \dots + \xi_n)^+)}) \leq K(c)e^{h(nc)} \quad (4.9)$$

holds for all n .

2. If $\mu \leq 0$, then an inequality corresponding to (4.9) is valid for any $c > 0$.

Proof. Let the function h be continued to negative real axis in a concave way. Denote $S_n := \xi_1 + \dots + \xi_n$. Suppose first that ξ has bounded left tail, i.e. $\xi \geq -M$ for some $M > 0$. Define $a := \mu/2$ and $\eta_n := \xi_n - c$ so that $\mathbb{E}(\eta_n) < 0$.

Now, for any $n \in \mathbb{N}$, $c > \mu$ and $b \geq 0$ it holds that

$$\begin{aligned} \mathbb{E}(e^{h(S_n+b)}) &= \mathbb{E}(e^{h(S_{n-1}+b+c+\eta_n)}) \\ &= \mathbb{E}(e^{h(S_{n-1}+b+c+\eta_n)} \mathbf{1}(S_{n-1} \leq a(n-1))) \\ &\quad + \mathbb{E}(e^{h(S_{n-1}+b+c+\eta_n)} \mathbf{1}(S_{n-1} > a(n-1))) \\ &\leq \mathbb{E}(e^{h(a(n-1)+b+c+\eta_n)} \mathbb{P}(S_{n-1} \leq a(n-1))) \\ &\quad + \mathbb{E}(e^{h(S_{n-1}+b+c+\eta_n)} \mathbf{1}(S_{n-1} > a(n-1))). \end{aligned}$$

Let x_0 be as in Lemma 4 of [5] and set $n_0 := \lceil x_0/a + 1 \rceil$. Now, for $n > n_0$

$$\mathbb{E}(e^{h(a(n-1)+b+c+\eta_n)}) \leq e^{h(a(n-1)+b+c)}.$$

Furthermore, by the same lemma,

$$\mathbb{E}(e^{h(S_{n-1}+b+c+\eta_n)} \mathbf{1}(S_{n-1} > a(n-1))) \leq \mathbb{E}(e^{h(S_{n-1}+b+c)}).$$

Hence,

$$\mathbb{E}(e^{h(S_n+b)}) \leq e^{h(a(n-1)+b+c)} \mathbb{P}(S_{n-1} \leq a(n-1)) + \mathbb{E}(e^{h(S_{n-1}+b+c)}).$$

Thus, inductive reasoning gives

$$\mathbb{E}(e^{h(S_n)}) \leq e^{h(cn)} \sum_{k=n_0}^{\infty} \mathbb{P}(S_k \leq ak) + \mathbb{E}(e^{h(S_{n_0}+nc)}). \quad (4.10)$$

The series on the right hand side of (4.10) converges because probabilities $\mathbb{P}(S_k \leq ak)$ decay at an exponential rate due to the fact that they are bounded from below and $a < \mu$. So, there is a constant $N(c)$ such that

$$\mathbb{E}(e^{h(S_n)}) \leq N(c)e^{h(cn)}. \quad (4.11)$$

In order to show (4.9) it suffices to note that

$$\mathbb{E}(e^{h((S_n)^+)}) \leq e^{h(0)} + \mathbb{E}(e^{h(S_n)})$$

and use (4.11).

For Part 1 it remains to consider the situation where ξ has unbounded left tail. Select $M_1 > 0$ so that

$$|\mathbb{E}(\xi \mathbf{1}(\xi < -M_1))| < (c - \mu)/2.$$

Then

$$\hat{\xi} := \xi \mathbf{1}(\xi \geq -M_1)$$

satisfies $\mathbb{E}(\hat{\xi}) =: \hat{\mu} \in (\mu, (\mu + c)/2)$. Now $\hat{\xi}$ has a bounded left tail and therefore the previous proof can be applied. Denote $\hat{S}_n := \hat{\xi}_1 + \dots + \hat{\xi}_n$ to get

$$\mathbb{E}(e^{h((S_n)^+)}) \leq \mathbb{E}(e^{h((\hat{S}_n)^+)}) \leq K(c)e^{h(nc)},$$

as required.

To show Case 2 one can estimate $\xi \leq \xi - \mu + c/2$ and apply Part 1 to random variable $\hat{\xi} := \xi - \mu + c/2$ with $\mathbb{E}(\hat{\xi}) = c/2$ where $c > 0$ is arbitrary. $\square$

Proof. (Proof of Theorem 2.2)

1. Recall that c_1 is a finite constant such that $c_1 > \mu > 0$. Note first that Assumption (2.5) implies

$$\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{h_X(x)} \geq \left(\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{h_N(x/c_1)} \right) \left(\liminf_{x \rightarrow \infty} \frac{h_N(x/c_1)}{h_X(x)} \right) \geq 1.$$

Using Part 2 of Lemma 1.1 we see that

$$\mathbb{E}(e^{(1-\epsilon)h_X(c_1 N)}) < \infty \quad (4.12)$$

for any $\epsilon > 0$.

Recall from assumptions that $\delta > 0$ is fixed. Set

$$0 < \eta < \delta/(1 + \delta).$$

Using the fact that

$$(1 - \eta)h_X(x) \geq \log x$$

for large enough x and Lemma 4.3 is applicable by definitions, we obtain the upper bound

$$\mathbb{E}(e^{(1-\eta)h_X(S_n^+)}) \leq K(c_1)e^{(1-\eta)h_X(c_1 n)} \quad (4.13)$$

for all $n \in \mathbb{N}$, where the constant $K(c_1)$ only depends on c_1 but not on n .

Utilising estimates (4.13) and (4.12) we get by (4.12)

$$\begin{aligned} \mathbb{E}(e^{(1-\eta)h_X(S_N^+)}) &= \sum_{k=1}^{\infty} \mathbb{E}(e^{(1-\eta)h_X(S_k^+)}) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} K(c_1)e^{(1-\eta)h_X(c_1 k)} \mathbb{P}(N = k) \\ &= K(c_1)\mathbb{E}(e^{(1-\eta)h_X(c_1 N)}) < \infty. \end{aligned}$$

Since the mapping $s \mapsto \mathbb{E}(e^{sh_X(S_N^+)})$ is increasing, $\mathbb{E}(e^{sh_X(S_N^+)}) < \infty$ holds for all $0 < s < 1$. Thus $\mathbb{I}_{h_X}(S_N) \geq 1$. On the other hand, for any $\epsilon > 0$ it holds that

$$\mathbb{E}(e^{(1+\epsilon)h_X(S_N^+)}) \geq \mathbb{E}(e^{(1+\epsilon)h_X(X^+)}) \mathbb{P}(N = m) \mathbb{P}(X \geq 0)^{m-1} = \infty$$

for some m and so $\mathbb{I}_{h_X}(S_N) \leq 1$. This proves that $\mathbb{I}_{h_X}(S_N) = 1$ and so h_X is a lower concave approximation of S_N .

2. Suppose $0 < \eta_1 < \delta/(1+\delta)$ is fixed and denote $\mathbb{E}(X) = \mu$. Fix the version of lower concave approximation of μN by setting $h_{\mu N}(x) := h_N(x/\mu)$. The mapping $c \mapsto \mathbb{I}_{h_{\mu N}}(cN)$ is continuous in a neighbourhood of μ , because by Lemma 1.1 $\mathbb{I}_{h_{\mu N}}(cN) = \mathbb{I}_{h_N}((c/\mu)N)$ and the latter is continuous at point 1 by Lemma 4.2.

Set $g(x) = h_N(\mu/x)$. Since $\mathbb{E}(e^{(1-\eta_1)h_N(N)}) < \infty$ for all $\eta_1 \in (0, 1)$, it holds that $1 - \eta_1 < \mathbb{I}_g(\mu N)$. In particular, by continuity we may find a number $c_{\eta_1} > \mu$ satisfying $1 - \eta_1 < \mathbb{I}_g(c_{\eta_1} N)$ so that

$$\mathbb{E}(e^{(1-\eta_1)h_{\mu N}(c_{\eta_1} N)}) < \infty. \quad (4.14)$$

Furthermore, we get

$$\liminf_{x \rightarrow \infty} \frac{R_X(x)}{h_{\mu N}(x)} \geq \left(\liminf_{x \rightarrow \infty} \frac{R_X(x)}{h_X(x)} \right) \left(\liminf_{x \rightarrow \infty} \frac{h_X(x)}{h_N(x/\mu)} \right) \geq 1$$

from Assumption (2.6) and thus $\mathbb{I}_g(X) \geq 1$ so that

$$\mathbb{E}(e^{(1-\eta_1)h_{\mu N}(X)}) < \infty. \quad (4.15)$$

In addition,

$$(1 - \eta_1)h_{\mu N}(x) = (1 - \eta_1)h_N(x/\mu) \geq \log x$$

for large enough x by the choice of η_1 . This makes Lemma 4.3 applicable. Using similar arguments as in the first part of the theorem and Equation (4.14) we obtain

$$\begin{aligned} \mathbb{E}(e^{(1-\eta_1)h_{\mu N}(S_N^+)}) &= \sum_{k=1}^{\infty} \mathbb{E}(e^{(1-\eta_1)h_{\mu N}(S_k^+)}) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} K(c_{\eta_1}) e^{(1-\eta_1)h_{\mu N}(c_{\eta_1} k)} \mathbb{P}(N = k) \\ &= K(c_{\eta_1}) \mathbb{E}(e^{(1-\eta_1)h_{\mu N}(c_{\eta_1} N)}) < \infty. \end{aligned}$$

This shows $\mathbb{I}_{h_{\mu N}}(S_N) \geq 1$.

Other direction follows from the law of large numbers: Assume $\eta_2 > 0$ is given. Then by Lemma 4.2 we may choose a number $0 < c_{\eta_2} < \mu$ so that

$$\mathbb{E}(e^{(1+\eta_2)h_{\mu N}(c_{\eta_2} N)}) = \infty. \quad (4.16)$$

In addition, by the law of large numbers we may find a constant $C =$

$C_{\eta_2, c_{\eta_2}} > 0$ such that $\mathbb{P}(S_k > c_{\eta_2} k) \geq C$ for all $k \in \mathbb{N}$. Now

$$\begin{aligned}
\mathbb{E}(e^{(1+\eta_2)h_{\mu N}(S_N^+)}) &= \sum_{k=1}^{\infty} \mathbb{E}(e^{(1+\eta_2)h_{\mu N}(S_k^+)}) \mathbb{P}(N = k) \\
&\geq \sum_{k=1}^{\infty} \mathbb{E}(e^{(1+\eta_2)h_{\mu N}(S_k^+)}) \mathbf{1}(S_k > c_{\eta_2} k) \mathbb{P}(N = k) \\
&= \sum_{k=1}^{\infty} e^{(1+\eta_2)h_{\mu N}(c_{\eta_2} k)} \mathbb{P}(S_k > c_{\eta_2} k) \mathbb{P}(N = k) \\
&\geq C \sum_{k=1}^{\infty} e^{(1+\eta_2)h_{\mu N}(c_{\eta_2} k)} \mathbb{P}(N = k) \\
&= C \mathbb{E}(e^{(1+\eta_2)h_{\mu N}(c_{\eta_2} N)}) = \infty.
\end{aligned}$$

This proves $\mathbb{I}_{h_{\mu N}}(S_N) \leq 1$ and we get $\mathbb{I}_{h_{\mu N}}(S_N) = 1$ so that $h_N(x/\mu)$ is a lower concave approximation of S_N . □

Remark 4.1. As it was kindly pointed out by a reviewer, some of the steps in the proof of Theorem 2.2 admit shorter alternative justifications based on well-known inequalities in the case where the increment dominates the number of claims. This is specifically true if it is assumed, in addition to the made assumptions, that the increments belong to the strongly subexponential class of heavy-tailed variables or that the increments can be stochastically bounded by variables of such class. In more detail, if Increment X_1 can be stochastically bounded by Variable Y_1 whose tail function is of the form $e^{-(1-\epsilon)h_X(x)}$ for large enough x , it follows from Theorem 3.30 of [8] that the distribution of Y_1 is strongly subexponential and consequently the known results for this distribution class become available for the estimates.

Proof. (Proof of Corollary 2.2.1) Note first that $\mathbb{P}(S_N > x) \geq \mathbb{P}(X > x)$ and therefore $R_{S_N}(x) \leq R_X(x)$ holds for all x , since $X \geq 0$ almost surely. Application of Theorem 2.2 and Assumption (2.7) gives

$$1 = \liminf_{x \rightarrow \infty} \frac{R_{S_N}(x)}{h_X(x)} \leq \limsup_{x \rightarrow \infty} \frac{R_{S_N}(x)}{h_X(x)} \leq \limsup_{x \rightarrow \infty} \frac{R_X(x)}{h_X(x)} = 1$$

and proves the claim. The proof of the remaining case is similar. □

Proof. (Proof of Lemma 2.3) Implication is verified by observing that (2.11) together with the definition of lower concave approximation and positivity yields

$$\liminf_{x \rightarrow \infty} \frac{R(x)}{\sqrt{x}} \geq \left(\liminf_{x \rightarrow \infty} \frac{h(x)}{\sqrt{x}} \right) \left(\liminf_{x \rightarrow \infty} \frac{R(x)}{h(x)} \right) > 0.$$

Hence, by Part 2 of Lemma 1.1, there exists $c > 0$ such that

$$\mathbb{E} \left(e^{c\sqrt{X}} \right) < \infty$$

and Theorem 1 of [23] confirms that X is determined by its moments. $\square$

Proof. (Proof of Theorem 2.4) The proof utilises similar techniques as the proof of Theorem 2.2.

1. Suppose first that Condition 1 of Theorem 2.4 holds. Recall that $c_1 > \mathbb{E}(X) > 0$ and $X \geq 0$. Since

$$\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{h_X(x)} \geq \left(\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{h_N(x/c_1)} \right) \left(\liminf_{x \rightarrow \infty} \frac{h_N(x/c_1)}{h_X(x)} \right) > 0,$$

there must exist $\eta > 0$ so that

$$\mathbb{E}(e^{\eta h_X(c_1 N)}) < \infty.$$

Clearly, by assumption (2.12), $\eta h_X(x) \geq \log x$ holds for all large enough x . This enables the use of Lemma 4.3.

We get

$$\begin{aligned} \mathbb{E}(e^{\eta h_X(S_N)}) &= \sum_{k=1}^{\infty} \mathbb{E}(e^{\eta h_X(S_k)}) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} K(c_1) e^{\eta h_X(c_1 k)} \mathbb{P}(N = k) \\ &= K(c_1) \mathbb{E}(e^{\eta h_X(c_1 N)}) < \infty. \end{aligned}$$

Hence $\mathbb{I}_{h_X}(S_N) \geq \eta > 0$. This implies, using Part 2 of Lemma 1.1, that

$$\liminf_{x \rightarrow \infty} \frac{R_{S_N}(x)}{h_X(x)} > 0. \quad (4.17)$$

By assumption (2.12) there exist a constant $c > 0$ such that $h_X(x) \geq c\sqrt{x}$ for all x large enough. Now (4.17) implies

$$\liminf_{x \rightarrow \infty} \frac{R_{S_N}(x)}{\sqrt{x}} > 0.$$

Hence $\sqrt{S_N}$ is light-tailed and by Hardy's criterion S_N is determined by its moments.

2. The proof of the second part follows same kind of arguments as the first part. First, we see that

$$\liminf_{x \rightarrow \infty} \frac{R_X(x)}{h_N(x/\mu)} \geq \left(\liminf_{x \rightarrow \infty} \frac{R_X(x)}{h_X(x)} \right) \left(\liminf_{x \rightarrow \infty} \frac{h_X(x)}{h_N(x/\mu)} \right) > 0,$$

ensuring the existence of small $\eta > 0$ so that

$$\mathbb{E}(e^{\eta h_N(X/\mu)}) < \infty.$$

Using Lemma 4.2 we may find a constant $c_\eta > \mu$ so that also

$$\mathbb{E}(e^{\eta h_N(c_\eta N/\mu)}) < \infty$$

holds. Now, again, by Lemma 4.3 we get

$$\begin{aligned} \mathbb{E}(e^{\eta h_N(S_N/\mu)}) &= \sum_{k=1}^{\infty} \mathbb{E}(e^{\eta h_N(S_k/\mu)}) \mathbb{P}(N = k) \\ &\leq \sum_{k=1}^{\infty} K(c_\eta) e^{\eta h_N(c_\eta k/\mu)} \mathbb{P}(N = k) \\ &= K(c_1) \mathbb{E}(e^{\eta h_N(c_\eta N/\mu)}) < \infty. \end{aligned}$$

The conclusion follows in the same way as in the first part.

□

A Technical Details

Technical details omitted during the main text are presented here.

A.1 Part 4 of Lemma 1.1

Proof of Part 4 of Lemma 1.1: Inequality $f_1 \geq h_X$. Proof of the first assertion can be justified using Theorem 2.10 of [8]. Let g be a non-negative, strictly increasing and continuous function for which $g(x) = o(R(x))$ and $g(x) \rightarrow \infty$, as $x \rightarrow \infty$. Thus, by Part 2 of Lemma 1.1 it holds that

$$\sup\{s \geq 0 : \mathbb{E}(e^{sg(X)}) < \infty\} = \liminf_{x \rightarrow \infty} \frac{R(x)}{g(x)} = \infty. \quad (\text{A.1})$$

Application of Theorem 2.10 of [8] shows that there exists a concave function, say $\hat{h}$, satisfying $\hat{h} \leq f_1$,

$$\mathbb{E}(e^{\hat{h}(X)}) < \infty \quad (\text{A.2})$$

and

$$\mathbb{E}(e^{\hat{h}(X)+g(X)}) = \infty. \quad (\text{A.3})$$

Without loss of generality we may additionally assume $\hat{h}(0) = 0$.

Now, for any $\epsilon \in (0, 1)$ it holds that

$$\mathbb{E}(e^{(1-\epsilon)\hat{h}(X)}) < \infty \quad (\text{A.4})$$

due to (A.2). It must also hold that

$$\mathbb{E}(e^{(1+\epsilon)\hat{h}(X)}) = \infty. \quad (\text{A.5})$$

If (A.5) was violated, (A.1) and Hölder's inequality for $1/p + 1/q = 1$, where $p = 1 + \epsilon$, would imply

$$\mathbb{E}(e^{\hat{h}(X)+g(X)}) \leq \mathbb{E}(e^{(1+\epsilon)\hat{h}(X)})^{1/(1+\epsilon)} \mathbb{E}(e^{qg(X)})^{1/q} < \infty$$

contradicting (A.3). Thus we may choose $h_X = \hat{h}$. $\square$

Proof of Part 4 of Lemma 1.1: Inequality $h_X^ \geq f_2$.* If $\sup\{s \geq 0 : \mathbb{E}(e^{sf_2(X)}) < \infty\} =: \alpha \in [1, \infty)$, one can take $h_X^*(x) = \alpha f_2(x)$.

Suppose that $\sup\{s \geq 0 : \mathbb{E}(e^{sf_2(X)}) < \infty\} = \infty$. Now, Part 2 of Lemma 1.1 implies

$$\lim_{x \rightarrow \infty} \frac{R(x)}{f_2(x)} = \infty.$$

Specifically, $2f_2(x) = o(R(x))$, as $x \rightarrow \infty$.

Let h be some lower concave approximation of X with $h(0) = 0$. Clearly,

$$\liminf_{x \rightarrow \infty} \frac{R(x)}{h(x) + 2f_2(x)} \leq \liminf_{x \rightarrow \infty} \frac{R(x)}{h(x)} = 1.$$

On the other hand, for any x large enough, $2f_2(x)/R(x) \leq 1$ holds. Hence

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{R(x)}{h(x) + 2f_2(x)} &= \liminf_{x \rightarrow \infty} \frac{1}{h(x)/R(x) + 2f_2(x)/R(x)} \\ &\geq \liminf_{x \rightarrow \infty} \frac{1}{h(x)/R(x) + 1} \\ &= \liminf_{x \rightarrow \infty} \frac{R(x)}{h(x) + R(x)} \\ &\geq \frac{1}{2} \liminf_{x \rightarrow \infty} \frac{R(x)}{\max(h(x), R(x))} \\ &= \frac{1}{2} \liminf_{x \rightarrow \infty} \frac{1}{\max(h(x)/R(x), 1)} \\ &= \frac{1}{2} \liminf_{x \rightarrow \infty} \left(\min \left(\frac{R(x)}{h(x)}, 1 \right) \right) = \frac{1}{2}, \end{aligned}$$

that is,

$$\liminf_{x \rightarrow \infty} \frac{R(x)}{h(x) + 2f_2(x)} =: c \in [1/2, 1].$$

In summary,

1.

$$\liminf_{x \rightarrow \infty} \frac{R(x)}{c(h(x) + 2f_2(x))} = 1,$$

2. $c(h(x) + 2f_2(x)) = 0$,

3. $c(h(x) + 2f_2(x))$ is concave as a sum of two concave functions

$$4. \ c(h(x) + 2f_2(x)) \geq 2cf_2(x) \geq f_2(x).$$

Since Properties 1-4 hold, setting $h_X^*(x) := c(h(x) + 2f_2(x))$ proves the claim. $\square$

A.2 Light-tailed vs. Heavy-tailed: Details for Remark 2.2

Additional results can be obtained in a straightforward way if the assumptions of Theorem 2.2 are modified.

1. Suppose N is light-tailed and X is heavy-tailed. Then the result of Part 1 of Theorem 2.2 holds when Assumption (2.5) is omitted. The proof of the result simplifies substantially compared to the case where both of the variables are heavy-tailed.

Firstly, since N is light-tailed, we have that $\mathbb{E}(e^{cN}) < \infty$ for some $c > 0$. Thus, by Lemma 1.1 Part 2 and the fact that $h_X(x) = o(x)$:

$$\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{h_X(x)} \geq \left(\liminf_{x \rightarrow \infty} \frac{R_{c_1 N}(x)}{x} \right) \left(\liminf_{x \rightarrow \infty} \frac{x}{h_X(x)} \right) = \infty.$$

This confirms that (4.12) holds and the rest of the proof is similar to the heavy-tailed case.

2. Suppose X is light-tailed and N is heavy-tailed. Then a similar modification as in the first part yields the result without Assumption (2.6) of Part 2 of Theorem 2.2.

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