WEIGHTED VECTOR-VALUED INEQUALITIES FOR A CLASS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, some weighted vector-valued inequalities with multiple weights \( A^\beta(L^{mn}) \) are established for a class of multilinear singular integral operators. The weighted estimates for the multi(sub)linear maximal operators which control the multilinear singular integral operators are also considered.

1. Introduction

In recent years, considerable attention has been paid to the boundedness of the multilinear singular integral operator on function spaces. Let \( K(x; y_1, \ldots, y_m) \) be a locally integrable function defined away from the diagonal \( x = y_1 = \cdots = y_m \) in \( \mathbb{R}^{mn} \). An operator \( T \) defined on \( \mathcal{S} \) (Schwartz space) and taking values in \( \mathcal{S}' \) is said to be an \( m \)-multilinear singular integral operator with kernel \( K \), if \( T \) is \( m \)-multilinear, and satisfies that

\[
T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x; y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy,
\]

for bounded functions \( f_1, \ldots, f_m \) with compact supports, and \( x \in \mathbb{R}^n \setminus \bigcap_{j=1}^m \text{supp} f_j \). Operators of this type were originated in the remarkable works of Coifman and Meyer [2], [3], and are useful in multilinear analysis. We say that \( K \) is a multilinear Calderón-Zygmund kernel, if \( K \) satisfies the size condition that for all \( (x, y_1, \ldots, y_m) \in \mathbb{R}^{(m+1)n} \) with \( x \neq y_j \) for some \( 1 \leq j \leq m \),

\[
|K(x; y_1, \ldots, y_m)| \lesssim \frac{1}{\left( \sum_{j=1}^m |x - y_j| \right)^{mn}}
\]

and satisfies the regularity condition that for some \( \beta \in (0, 1] \)

\[
|K(x; y_1, \ldots, y_m) - K(x'; y_1, \ldots, y_m)| \lesssim \frac{|x - x'|^\beta}{\left( \sum_{j=1}^m |x - y_j| \right)^{mn+\beta}}
\]

whenever \( \max_{1 \leq k \leq m} |x - y_k| \geq 2|x - x'| \), and for all \( 1 \leq j \leq m \),

\[
|K(x; y_1, \ldots, y_j, \ldots, y_m) - K(x; y_1, \ldots, y_j', \ldots, y_m)| \lesssim \frac{|y_j - y_j'|^\beta}{\left( \sum_{i=1}^m |x - y_i| \right)^{mn+\beta}}
\]

whenever \( \max_{1 \leq k \leq m} |x - y_k| \geq 2|y_j - y_j'| \). When \( K \) is a multilinear Calderón-Zygmund kernel, Grafakos and Torres [12] considered the behavior of \( T \) on \( L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \), and established a \( T1 \) type theorem for the operator \( T \). Lerner, Ombrosi, Pérez, Torres and Trojillo-Gonzalez [16] introduced a new maximal operator and a new class of multiple weights \( A^\beta(L^{mn}) \) (see Definition 1.9 below), and established the weighted estimates with \( A^\beta(L^{mn}) \) for the multilinear Calderón-Zygmund

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A family of operators $\{A_t\}_{t > 0}$ is said to be an approximation to the identity, if for every $t > 0$, $A_t$ can be represented by the kernel at in the following sense: for every function $u \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty]$ and almost everywhere $x \in \mathbb{R}^n$,

$$A_t u(x) = \int_{\mathbb{R}^n} a_t(x, y) u(y) dy,$$

and the kernel $a_t$ satisfies that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|a_t(x, y)| \leq h_t(x, y) = t^{-n/s} \phi \left( \frac{|x - y|}{t^{1/s}} \right),$$

where $s > 0$ is a constant and $h$ is a positive, bounded and decreasing function such that for some constant $\eta > 0$,

$$\lim_{r \to \infty} r^{n+\eta} h(r^s) = 0.$$

Assumption 1.6. For each fixed $j$ with $1 \leq j \leq m$, there exists an approximation to the identity $\{A_t^j\}_{t > 0}$ with kernels $\{a_t^j(x, y)\}_{t > 0}$, and there exist kernels $K_t^j(x; y_1, \ldots, y_m)$, such that for bounded functions $f_1, \ldots, f_m$ with compact supports, and $x \in \mathbb{R}^n \cap \cap_{k=1}^m \text{supp} f_k$,

$$T(f_1, \ldots, f_{j-1}, A_t^j f_j, f_{j+1}, \ldots, f_m)(x) = \int_{\mathbb{R}^m} K_t^j(x; y_1, \ldots, y_m) \prod_{k=1}^m f_k(y_k) dy,$$

and there exists a function $\phi \in C(\mathbb{R})$ with support $\phi \subset [-1, 1]$, and a constant $\varepsilon \in (0, 1]$, such that for all $x, y_1, \ldots, y_m \in \mathbb{R}^n$ and all $t > 0$ with $2t^{1/s} \leq |x - y_1|$, \[ |K(x; y_1, \ldots, y_m) - K_t^j(x; y_1, \ldots, y_m)| \leq \frac{t^{s/2}}{\left( \sum_{k=1}^m |x - y_k| \right)^{mn+\varepsilon}} + \frac{1}{\left( \sum_{k=1}^m |x - y_k| \right)^{mn}} \sum_{1 \leq i \leq m, i \neq j} \phi \left( \frac{|y_i - y_j|}{t^{1/s}} \right). \]

As it was pointed out in [6], operators with such kernels are called multilinear singular integral operators with non-smooth kernels, since the kernel $K$ satisfying Assumption 1.6 may enjoy no smoothness in the variables $y_1, \ldots, y_m$. Duong, Grafakos and Yan proved that if $T$ satisfies Assumption 1.6, and is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q, \infty}(\mathbb{R}^n)$ for some $q_1, \ldots, q_m \in (1, \infty)$ and $q \in (0, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_m$, then $T$ is also bounded from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{1/m, \infty}(\mathbb{R}^n)$. Let $T^*$ be the maximal operator associated with the operator $T$ satisfying Assumption 1.6, that is,

$$T^*(f_1, \ldots, f_m)(x) = \sup_{\epsilon > 0} \left[ \int_{\mathbb{R}^n} \sum_{j=1}^m |x - y_j|^2 > \epsilon^2 K(x; y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy \right].$$

To consider the weighted estimates with $A_p(\mathbb{R}^n)$ weights for $T^*$, Duong et al. [5] introduced the following two assumptions.

Assumption 1.7. There exists an approximation to the identity $\{B_t\}_{t > 0}$ with kernels $\{b_t(x, y)\}_{t > 0}$, and there exist kernels $\{K_t^0(x; y_1, \ldots, y_m)\}_{t > 0}$ such that

$$K_t^0(x; y_1, \ldots, y_m) = \int_{\mathbb{R}^n} K(z; y_1, \ldots, y_m) b_t(x, z) dz.$$
and there exists a function \( \psi \in C(\mathbb{R}) \) with \( \text{supp} \psi \subset [-1, 1] \), and a constant \( \gamma \in (0, 1] \), such that for all \( x, y_1, \ldots, y_m \in \mathbb{R}^n \) and all \( t > 0 \) with \( 2t^{1/s} \leq \max_{1 \leq k \leq m} |x - y_k| \),
\[
|K(x; y_1, \ldots, y_m) - K_0^0(x; y_1, \ldots, y_m)| \\
\leq \frac{1}{(\sum_{k=1}^m |x - y_k|)^{\min(1, \gamma)}} + \frac{1}{(\sum_{k=1}^m |x - y_k|)^{\min(2, \gamma)}} \sum_{1 \leq j \leq m} \psi \left( \frac{|x - y_j|}{t^{1/s}} \right).
\]

**Assumption 1.8.** The kernel \( K_0^0(x; y_1, \ldots, y_m) \) in Assumption 1.7 satisfies the size condition that
\[
|K_0^0(x; y_1, \ldots, y_m)| \leq \frac{1}{(\sum_{j=1}^m |x - y_j|)^{mn}}
\]
whenever \( 2t^{1/s} \leq \min_{1 \leq j \leq m} |x - y_j| \), and the regularity condition that
\[
|K_0^0(x; y_1, \ldots, y_m) - K_0^0(x'; y_1, \ldots, y_m)| \leq \frac{t^{1/s}}{(\sum_{j=1}^m |x - y_j|)^{mn}}
\]
whenever \( 2|x - x'| \leq t^{1/s} \) and \( 2t^{1/s} \leq \min_{1 \leq j \leq m} |x - y_j| \).

Duong et al. [5] proved that if \( T \) satisfies Assumption 1.6, Assumption 1.7 and Assumption 1.8, and is bounded from \( L^p(\mathbb{R}^n) \times \cdots \times L^p(\mathbb{R}^n) \) to \( L^{q, \infty}(\mathbb{R}^m) \) for some \( q_1, \ldots, q_m \in (1, \infty) \) and \( q \in (0, \infty) \) with \( 1/q = 1/q_1 + \cdots + 1/q_m \), then for \( p_1, \ldots, p_m \in [1, \infty) \) and \( p \in (0, \infty) \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \), and \( w \in A_{\text{min}_{1 \leq j \leq m} p_j}(\mathbb{R}^n) \), both \( T \) and \( T^* \) are bounded from \( L^p(\mathbb{R}^n, w) \times \cdots \times L^p(\mathbb{R}^n, w) \) to \( L^{r, \infty}(\mathbb{R}^n, w) \), and when \( \min_{1 \leq j \leq m} p_j > 1 \), \( T \) and \( T^* \) are bounded from \( L^p(\mathbb{R}^n, w) \times \cdots \times L^p(\mathbb{R}^n, w) \) to \( L^p(\mathbb{R}^n, w) \). Grafakos, Liu and Yang [10] considered the weighted norm inequalities with multiple weights for \( T^* \), and proved that \( T \) and \( T^* \) enjoy the weighted estimates with \( A_w(\mathbb{R}^{mn}) \) weights the same as the multilinear Calderón-Zygmund operators.

The purpose of this paper is to establish some weighted vector-valued inequalities with multiple weights for a class of multilinear singular integral operators, as analogs of the weighted vector-valued inequalities with \( A_p(\mathbb{R}^n) \) weights for the classical Calderón-Zygmund operators (see [1]) in the setting of multilinear singular integral operators. We remark that the operators we consider here, contain the multilinear Calderón-Zygmund operators and the multilinear singular integral operators with non-smooth kernels as examples, see Remark 1.12 below. To state our results, we first recall some definitions and notations.

Let \( p, r \in (0, \infty) \) and \( w \) be a weight. As usual, for a sequence of numbers \( \{a_k\}_{k=1}^\infty \), we denote \( \|\{a_k\}\|_r = \left( \sum_k |a_k|^r \right)^{1/r} \). The space \( L^p(I^r; \mathbb{R}^n, w) \) is defined as
\[
L^p(I^r; \mathbb{R}^n, w) = \{ \{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} < \infty \}
\]
where
\[
\|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} \left( \int_{I^r} |f_k(x)|^p w(x) \, dx \right)^{1/p} \right)^{1/p}.
\]
The space \( L^{p, \infty}(I^r; \mathbb{R}^n, w) \) is defined as
\[
L^{p, \infty}(I^r; \mathbb{R}^n, w) = \{ \{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} < \infty \}
\]
with
\[
\|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} = \sup_{\lambda > 0} \lambda^p w \left( \left\{ x \in \mathbb{R}^n : \|\{f_k(x)\}\|_r > \lambda \right\} \right).
\]
When \( w \equiv 1 \), we denote \( \|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} \) (\( \|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} \)) by \( \|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n)} \) (\( \|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n)} \)) for simplicity.

The following definition of multiple weights was introduced in [16].
Definition 1.9. Let \( m \in \mathbb{N} \), \( w_1, \ldots, w_m \) be weights, \( p_1, \ldots, p_m \in [1, \infty) \), \( p \in (0, \infty) \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \). Set \( \vec{w} = (w_1, \ldots, w_m) \), \( P = (p_1, \ldots, p_m) \) and \( \nu_{\vec{w}} = \prod_{k=1}^{m} w_k^{1/p_k} \). We say that \( \vec{w} \in A_\vec{p}(\mathbb{R}^m) \) if

\[
\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \nu_{\vec{w}}(x) \, dx \right)^{1/p} \prod_{k=1}^{m} \left( \frac{1}{|Q|} \int_Q w_k^{-\frac{1}{p_k}}(x) \, dx \right)^{1-1/p_k} < \infty,
\]

when \( p_k = 1 \), \( \left( \frac{1}{|Q|} \int_Q w_k^{-\frac{1}{p}}(x) \, dx \right)^{1-1/p_k} \) is understood as \((\inf_{Q} w_k)^{-1}\).

Our first result can be stated as follows.

Theorem 1.10. Let \( m \geq 2 \), \( T \) be an \( m \)-linear operator with kernel \( K \) in the sense of (1.1), \( r_1, \ldots, r_m \in (1, \infty) \), \( r \in (0, \infty) \) such that \( 1/r = 1/r_1 + \cdots + 1/r_m \). Suppose that

\begin{enumerate}
  
  \item \( T \) is bounded from \( L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \);
  \item for \( x, x', y_1, \ldots, y_m \in \mathbb{R}^n \) with \( 8|x-x'| < \min_{1 \leq j \leq m} |x-y_j| \), and each number \( D \) such that \( 2|x-x'| < D \) and \( 4D < \min_{1 \leq j \leq m} |x-y_j| \),
  \item \( |K(x; y_1, \ldots, y_m) - K(x'; y_1, \ldots, y_m)| \lesssim \frac{D^n}{(\sum_{j=1}^{m} |x-y_j|)} \).
\end{enumerate}

Remark 1.12. As it was pointed out in [6], if \( T \) is an \( m \)-linear Calderón-Zygmund operator, then \( T \) satisfies Assumption 1.6. On the other hand, it was proved in [14] that, if \( T \) satisfies Assumptions 1.7 and 1.8, then \( K \) satisfies (1.6). This shows that, the multilinear singular integral operators considered in [6, 5] satisfy the hypothesis of Theorems 1.10.
with
\[
\mathcal{M}_i(f_1, \ldots, f_m)(x) = \sup_{Q \ni x} \sum_{i=1}^{\infty} 2^{-ni} \left( \frac{1}{|Q|} \int_Q |f_i(y)| \, dy \right) \times \prod_{1 \leq j \leq m} \left( \frac{1}{|2^{i}Q|} \int_{2^{i}Q} |f_j(y)| \, dy \right).
\]

For the operators $\mathcal{M}$ and $\mathcal{M}_i$, we have

**Theorem 1.13.** Let $p_1, \ldots, p_m \in [1, \infty)$, $q_1, \ldots, q_m \in (1, \infty)$, $p, q \in (0, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, $1/q = 1/q_1 + \cdots + 1/q_m$, $\vec{w} = (w_1, \ldots, w_m) \in A_p(\mathbb{R}^{mn})$. All of the operators $\mathcal{M}, \mathcal{M}_i$ for $i = 1, \ldots, m$ are bounded from $L^{p_1}(\mathbb{R}^n; \mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n; \mathbb{R}^n, w_m)$ to $L^{q_1}(\mathbb{R}^n; \mathbb{R}^n, \nu_{\vec{w}})$. Moreover, if $\min_{1 \leq j \leq m} p_j > 1$, then the operators $\mathcal{M}$ and $\mathcal{M}_i$ are bounded from $L^{p_1}(\mathbb{R}^n; \mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n; \mathbb{R}^n, w_m)$ to $L^{q}(\mathbb{R}^n; \mathbb{R}^n, \nu_{\vec{w}})$.

In what follows, $C$ always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. Constant with subscript such as $C_1$, does not change in different occurrences. For any set $E \subset \mathbb{R}^n$, $\chi_E$ denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ (diam$Q$) to denote the side length (diameter) of $Q$, and $\lambda Q$ to denote the cube with the same center as $Q$ and whose side length is $\lambda$ times that of $Q$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes the ball centered at $x$ and having radius $r$.

## 2. Proof of Theorem 1.10

We begin with a variant of the Whitney decomposition lemma, see [19].

**Lemma 2.1.** Let $R > 1$. There exists a constant $C(n, R)$ such that for all open set $\Omega \subset \mathbb{R}^n$, $\Omega$ can be decomposed as $\Omega = \bigcup_j Q_j$, where $\{Q_j\}$ is a sequence of cubes with disjoint interiors, and

(i) $5R \leq \text{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leq 15R$,

(ii) $\sum_j \chi_{RQ_j}(x) \leq C_n, R \chi_{\Omega}(x)$.

Let $f \in L^1(\mathbb{R}^n)$ and $Mf$ be the Hardy-Littlewood maximal function of $f$. Applying Lemma 2.1 to the set $\Omega = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$, we can obtain a sequence of cubes $\{Q_j\}$ with disjoint interiors, such that

$$
\frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, dy > \lambda.
$$

As in [20, p. 19], we can verify that, for each $j$, there exists a cube $Q'_j$ which contains a point $x_j$ such that $Mf(x_j) \leq \lambda$, $Q'_j \supset Q_j$, $\ell(Q'_j) = (15R + 1)n\ell(Q_j)$. Therefore,

$$
\frac{1}{|Q'_j|} \int_{Q'_j} |f(y)| \, dy \leq \frac{1}{|Q_j|} \int_{Q_j} |f(y)| \, dy \lesssim Mf(x_j) \lesssim \lambda.
$$

Moreover, $\sum_j \chi_{RQ'_j}(x) \lesssim \chi_{\Omega}(x)$.

**Lemma 2.2.** Let $m \geq 2$, $1 \leq j \leq m$, $T$ be an $m$-linear operator with kernel $K$ in the sense of (1.1), $q_1, \ldots, q_m \in (1, \infty)$ with $q \in (0, \infty)$ such that $1/q = 1/q_1 + \cdots + 1/q_m$. Suppose that

(i) $T$ is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q}(\mathbb{R}^n)$;
(ii) \( T \) satisfies the Assumption 1.6.

Then \( T \) is bounded from \( L^1(l^r_1, \mathbb{R}^n) \times \cdots \times L^1(l^r_m, \mathbb{R}^n) \) to \( L^{1/m, \infty}(l^r, \mathbb{R}^n) \).

Proof. We claim that if \( p_1, \ldots, p_m \in (1, \infty), r_1, \ldots, r_m \in (1, \infty), \) \( 1/p = 1/p_1 + \cdots + 1/p_m, \) \( 1/r = 1/r_1 + \cdots + 1/r_m \), then for each \( 1 \leq j_0 \leq m \), \( T \) is bounded from \( \prod_{i=1}^{m} L^{p_i}(l^{r_i}, \mathbb{R}^n) \) to \( L^{p, \infty}(l^{r}, \mathbb{R}^n) \), then for each \( 1 \leq j_0 \leq m \), \( T \) is bounded from \( \prod_{i=1}^{m} L^{p_i}(l^{r_i}, \mathbb{R}^n) \times \cdots \times L^{p_{j_0-1}}(l^{r_{j_0-1}}, \mathbb{R}^n) \times L^{1}(l^{r_{j_0}}, \mathbb{R}^n) \times L^{p_{j_0+1}}(l^{r_{j_0+1}}, \mathbb{R}^n) \times \cdots \times L^{p_m}(l^{r_m}, \mathbb{R}^n) \) to \( L^{p, \infty}(l^{r}, \mathbb{R}^n) \), where \( 1/q_{j_0} = \sum_{1 \leq j \leq m, j \neq j_0} 1/p_j + 1. \) In fact, this is equivalent to prove that for each fixed \( \lambda > 0 \),

\[
\left\{ x \in \mathbb{R}^n : \| T(f^1_m, \ldots, f^m_m)(x) \|_r > \lambda \right\} \leq \lambda^{-\theta_{j_0}} \prod_{1 \leq j \leq m, j \neq j_0} \| \{ f^1_j \} \|_{L^{p,j}(l^{r}, \mathbb{R}^n)} \| \{ f^m_j \} \|_{L^{1}(l^{r_{j_0}}, \mathbb{R}^n)}.\]

For simplicity, we only consider the case \( j_0 = m \). By homogeneity, we may assume that

\[
\| \{ f^1_m \} \|_{L^{p_1}(l^{r_1}, \mathbb{R}^n)} = \cdots = \| \{ f^m_m \} \|_{L^{p_m}(l^{r_m}, \mathbb{R}^n)} = 1.
\]

For \( \lambda > 0 \), applying Lemma 2.1 to \( \Omega_m = \{ x \in \mathbb{R}^n : M(\| f^m_m \|_{L^{r_m}})(x) > \lambda^{\epsilon_m} \} \) and \( R = 4 \), we obtain a sequence of cubes \( \{ Q_m^i \} \) with disjoint interiors, such that

\[
\lambda^{\epsilon_m} < \frac{1}{|Q_m^i|} \int_{Q_m^i} \| f^m_m(x) \|_{L^{r_m}} dx \lesssim \lambda^{-\epsilon_m},
\]

and \( \sum_i \chi_{Q_m^i}(x) \lesssim \chi_{\Omega_m}(x) \). For each fixed \( k \), set

\[
f^k_m(x) = f^k_m(x) \chi_{\mathbb{R}^n \setminus \Omega_m}(x),
\]

\[
f^2_m(x) = \sum_i A^n_{Q_m^i} b^{k,i}_m(x),
\]

\[
f^3_m(x) = \sum_i (b^{k,i}_m(x) - A^n_{Q_m^i} b^{k,i}_m(x)),
\]

with \( b^{k,i}_m(y) = f^k_m(y) \chi_{Q_m^i}(y) \). Our proof is now reduced to proving that for \( i = 1, 2, 3 \),

\[
\left\{ x \in \mathbb{R}^n : \| T(f^1_m, \ldots, f^{m-1}_m; f^k_m)(x) \|_{L^{r}} > \lambda^{3/4} \right\} \lesssim \lambda^{-\epsilon_m},
\]

We first prove (2.3) for \( i = 1, 2 \). By the fact that \( \| \{ f^k_m \} \|_{L^{1/m, \infty}(l^{r}, \mathbb{R}^n)} \lesssim \lambda^{\epsilon_m} \), we deduce that

\[
\| \{ f^k_m \} \|_{L^{p_{j_0-1}}(l^{r_{j_0-1}}, \mathbb{R}^n)} \lesssim \lambda^{\epsilon_m \frac{p_{j_0-1}}{p_{j_0}}}.\]

Recalling that \( T \) is bounded from \( L^{p_1}(l^{r_1}, \mathbb{R}^n) \times \cdots \times L^{p_m}(l^{r_m}, \mathbb{R}^n) \) to \( L^{p, \infty}(l^{r}, \mathbb{R}^n) \), and \( 1/q_{j_0} = \sum_{j=1}^{m} 1/p_j + 1 \), we have by the inequality (2.4) that

\[
\left\{ x \in \mathbb{R}^n : \| T(f^1_m, \ldots, f^{m-1}_m; f^k_m)(x) \|_{L^{r}} > \lambda^{3/4} \right\} \lesssim \lambda^{-\epsilon_m}.
\]

To prove (2.3) for \( i = 2, 3 \), we first get from (1.4) and (1.5) that

\[
\int_{\mathbb{R}^n} \left| v_h(y) A^n_{Q_m^j} b^{k,i}_m(y) \right| dy \leq \int_{Q_m^j} \left| b^{k,i}_m(z) \right| \int_{\mathbb{R}^n} b^{k,i}_m(z, y) |v_h(z)| dz dy \leq \int_{Q_m^j} \left| b^{k,i}_m(z) \right| dz \inf_{y \in Q_m^j} M v_h(y).
\]

On the other hand, a straightforward computation involving Minkowski’s inequality gives us that

\[
\left( \sum_k \| b^{k,i}_m \|_{L^{r_j}(l^{r_j})} \right)^{1/r_j} \leq \int_{Q_m^j} \left( \sum_k \| f^k_m \|_{L^{r_j}(l^{r_j})} \right)^{1/r_j} dy \lesssim \lambda^{\epsilon_m |Q_m^j|}.
\]
Therefore, by Minkowski’s inequality and the vector-valued inequality of the Hardy-Littlewood maximal operator $M$ (see [7]),

\[
\left\|\sum_k \left|\sum_l A_{Q_m}^{m} b_{k,l}^{m} y \right|^{r_m} \right|^{1/r_m}_{L^p(\mathbb{R}^n)} \leq \sup_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_k \sum_l \int_{\mathbb{R}^n} |v_k(y) A_{Q_m}^{m} b_{k,l}^{m}(y)| dy \\
\leq \sup_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_k \sum_l \int_{Q_m} |b_{k,l}^{m}(z)| dz \inf_{y \in Q_m} Mv_k(y) \\
\leq \sup_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_k \sum_l \int_{Q_m} \left\|b_{k,l}^{m}(z)\right\|_{L^p} \sum_l Mv_k(y) dy \\
\leq \lambda^m \sum_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_l \int_{Q_m} \left\|Mv_k(y)\right\|_{L^p} dy \\
\leq \lambda^m \sup_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_l \int_{Q_m} \left\|Mv_k(y)\right\|_{L^p} dy \\
\leq \lambda^m \sup_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_l \int_{Q_m} \left\|Mv_k(y)\right\|_{L^p} dy \\
\leq \lambda^m \sum_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_l \int_{Q_m} \left\|Mv_k(y)\right\|_{L^p} dy \\n\leq \lambda^m \sum_{\|v_k\|_{L^p(\mathbb{R}^n)}} \sum_l \int_{Q_m} \left\|Mv_k(y)\right\|_{L^p} dy \\
\leq \lambda^m \lambda^{m-1} \lambda^m. 
\]

This, along with the fact that $T$ is bounded from $L^{m}(I^r; \mathbb{R}^n) \times \cdots \times L^{m}(I^r; \mathbb{R}^n)$ to $L^{p, \infty}(I^r; \mathbb{R}^n)$, leads to that

\[
\left\{x \in \mathbb{R}^n : \left\|\left\{T(f^k_1, \ldots, f^k_{m-1}, f^k_m^2)(x)\right\}\right\|_{L^r} > \lambda/3\right\} \leq \lambda^{-\theta_m}.
\]

Now we prove the estimate (2.3) for $i = 3$. Let $\Omega_m = \cup_i 4nQ_m^i$. It is obvious that $|\Omega_m| \leq \lambda^{-\theta_m}$. Let

\[
I_k(x) = \sum_l \int_{\mathbb{R}^n} \frac{\left\{\ell(Q_m^i)\right\}^\varepsilon}{(\sum_{j=1}^{m} |x - y_j|)^{mn+\varepsilon}} \prod_{j=1}^{m-1} |f^k_j(y_j)| |b_{k,l}^{m}(y_m)| dy, \\

\]

\[
\Pi_k(x) = \sum_{j=1}^{m-1} \sum_l \int_{\mathbb{R}^n} \phi\left(\frac{|y_j - y_m|}{\ell(Q_m^i)}\right) \prod_{i=1}^{m} |f^k_i(y_i)| |b_{k,l}^{m}(y_m)| dy.
\]

By Assumption 1.6, we know that for each $x \in \mathbb{R}^n \setminus \Omega_m$,

\[
|T(f^k_1, \ldots, f^k_{m-1}, f^k_m^2)(x)| \leq \sum_l \int_{\mathbb{R}^n} |K(x; y_1, \ldots, y_m) - K_{A_{Q_m^i}}^m(x; y_1, \ldots, y_m) \prod_{j=1}^{m-1} |f^k_j(y_j)| |b_{k,l}^{m}(y_m)| dy \\
\leq I_k(x) + \Pi_k(x).
\]

Let $\chi_m$ be the center of $Q_m^i$, and

\[
\chi_m^k(x) = \sum_l \frac{\left\{\ell(Q_m^i)\right\}^\varepsilon}{|x - x_m|^{n+\varepsilon}} |b_{k,l}^{m}|_{L^1(\mathbb{R}^n)}.
\]

It follows from (2.5) that

\[
\left\|\chi_m^k(x)\right\|_{L^p(\mathbb{R}^n)} \leq \lambda^m \sum_{l} \left(\sum_k |b_{k,l}^{m}|_{L^p(\mathbb{R}^n)} \right)^1/r_m \left\{\ell(Q_m^i)\right\}^\varepsilon \frac{|x - x_m|^{n+\varepsilon}}{|x - x_m|^{n+\varepsilon}},
\]

\[
\leq \lambda^m \sum_{l} |Q_m^i| \left\{\ell(Q_m^i)\right\}^\varepsilon |x - x_m|^{n+\varepsilon}.
\]
Let again by Minkowski’s inequality, we then deduce that
\[ \mu \]

On the other hand, it follows from (2.8)

Our goal is to prove that for each \( j \)

Observing that
\[ \nu \]

We turn our attention to
\[ \Phi \]

We now prove (2.7).

We consider the following two cases.

Case I. \( p_j = 1 \). For \( x \in \mathbb{R}^n \), write

\[ \Pi_j(x) = \sum_{I} \int_{\mathbb{R}^n} \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^m} \phi \left( \frac{|y_j - y_m|}{\ell(Q_m^j)} \right) \prod_{i=1}^{m-1} |f_j^k(y_i)||h_{m,j}^k(y_m)| dy_j dy_m \]

Our goal is to prove that for each \( j \) with \( 1 \leq j \leq m-1 \),

(2.8) \[ \left\{ x \in \mathbb{R}^n \mid \Pi_j(x) > \lambda/(6m) \right\} \leq \chi = \chi^m. \]

If this is true, then (2.3) with \( i = 3 \) follows from (2.6), (2.7) and (2.8) directly.

We now prove (2.8). We consider the following two cases.

Case I. \( p_j = 1 \). For \( x \in \mathbb{R}^n \), write

\[ \Pi_j(x) = \prod_{1 \leq j \leq m-1, \, j \neq k} Mf_j^k(x) \sum_{I} \int_{\mathbb{R}^n} \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^m} \phi \left( \frac{|y_j - y_m|}{\ell(Q_m^j)} \right) \prod_{i=1}^{m-1} |f_j^k(y_i)||h_{m,j}^k(y_m)| dy_j dy_m \]

Let

\[ D_{j,m}^k(x) = \int_{Q_m^j} |f_j^k(y_j)| dy_j, \quad E_{j,m}^k(x) = \frac{||h_{m,j}^k||_{L_1(\mathbb{R}^n)}}{|x - x_m|^2}, \]

Again by Minkowski’s inequality,

(2.9) \[ \left\{ \sum_k (D_{j,m}^k(x))^{r_j} \right\}^{1/r_j} \leq \int_{Q_m^j} \left\{ \sum_k |f_j^k(y_j)| \right\}^{r_j} dy_j. \]

On the other hand, it follows from (2.5) that

(2.10) \[ \left\{ \sum_k (E_{j,m}^k(x))^{r_m} \right\}^{1/r_m} \leq \chi = \chi^m. \]

Set \( \mu_j, m \in (0, \infty) \) such that \( 1/\mu_j, m = 1/r_j + 1/r_m \). We can take \( \nu \in (1/2, 1) \) such that \( \nu_j, m/\mu > 1 \) since \( \mu_j, m > 1/2 \). Let

\[ F_j^m(x) = \left\{ \sum_l \frac{|Q_m^j|^\mu}{|x - x_m|^{2\mu}} \left[ \int_{Q_m^j} \left\{ \sum_k |f_j^k(y_j)| \right\}^{r_j} dy_j \right]^{\mu} \right\}^{1/\nu}. \]
Thus, by Hölder’s inequality, (2.9) and (2.10), now tells us that

\[
\left( \sum_k \left| \sum_l E_{x,m,k}^l (x) D_{x,m,k}^l (x) \right|^{\mu_j, m} \right)^{\frac{1}{\mu_j, m}} \leq \left( \left\{ \sum_k \left[ \sum_l \left| E_{x,m,k}^l (x) D_{x,m,k}^l (x) \right|^{\mu_j, m} \right] \right\}^\frac{\mu_j, m}{\mu_j, m} \right) \frac{1}{\mu_j, m} \]

\[
\lesssim \left\{ \sum_l \left( \sum_k \left| E_{x,m,k}^l (x) \right|^{\lambda, m} \left( \sum_k \left| D_{x,m,k}^l (x) \right|^{r_m} \right)^{\frac{1}{r_m}} \right) \right\} \frac{1}{\mu_j, m} \]

\[
\lesssim \lambda^{\theta m} F_j^m (x),
\]

Thus, by Hölder’s inequality,

\[
\| \{ \Pi_k^l (x) \} \|_{r_j} \lesssim \lambda^{\theta m} F_j^m (x) \prod_{1 \leq i \leq m-1, i \neq j} \| \{ M f_j^k (x) \} \|_{r_j},
\]

It is easy to verify that

\[
\int_{\mathbb{R}^n \setminus \Omega_m} |F_j^m (x)|^{\mu_j} \, dx \lesssim \sum_l |Q_m^l|^{-\mu_j + 1} \left( \int_{4Q_m^l} \| \{ f_j^k (y_j) \} \|_{r_j} \, dy_j \right)^{\mu_j}
\]

\[
\lesssim \left( \sum_l |Q_m^l|^{-\mu_j} \left( \sum_l \int_{4Q_m^l} \| \{ f_j^k (y_j) \} \|_{r_j} \, dy_j \right)^{\mu_j} \right)
\]

\[
\lesssim \lambda^{-\theta m \mu_j} \| \{ f_j^k \} \|_{L^1 (r_j, \mathbb{R}^n)}.
\]

Recall that \( p_j = 1 \). We obtain from (2.11) and (2.12) and the vector-valued inequality for the operator \( M \) that

\[
\left| \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega}_m : \| \{ \Pi_k^l (x) \} \|_{r_j} > \lambda/(6m) \right\} \right| \lesssim \lambda^{-\theta m} \int_{\mathbb{R}^n \setminus \tilde{\Omega}_m} |F_j^m (x)|^{\mu_j} \, dx
\]

\[
+ \sum_{1 \leq i \leq m, i \neq j} \left| \left\{ x \in \mathbb{R}^n : \| \{ M f_j^k (x) \} \|_{r_j} > \lambda^{\theta m} \right\} \right| \lesssim \lambda^{-\theta m}.
\]

**Case II** \( p_j \in (1, \infty) \). We take \( \sigma \in (1, \min \{ p_j, r_j \}) \). Set

\[
G(x) = \sum_l \frac{|Q_m^l|^{2-1/\sigma}}{|x - x_m^l|^{2n-\sigma/\sigma}}.
\]

It is easy to verify that

\[
\int_{\mathbb{R}^n \setminus \tilde{\Omega}_m} G(x) \, dx \lesssim \sum_l |Q_m^l| \lesssim \lambda^{-\theta m}.
\]

For \( x \in \mathbb{R}^n \setminus \tilde{\Omega}_m \), it is obvious that \( 4Q_m^l \subset B(x, 2|x - x_m^l|) \) and so we have

\[
\int_{4Q_m^l} |f_j^k (y_j)| \, dy_j \lesssim |Q_m^l|^{1-\frac{1}{\sigma}} \left( \int_{4Q_m^l} |f_j^k (y_j)|^\sigma \, dy_j \right)^{\frac{1}{\sigma}}
\]

\[
\lesssim |Q_m^l|^{1-\frac{1}{\sigma}} |x - x_m^l|^{\frac{1}{\sigma}} M \sigma f_j^k (x),
\]

where and in the following, \( M \sigma f(x) = [M (f^\sigma) (x)]^{1/\sigma} \). It then follows from Hölder’s inequality that when \( x \in \mathbb{R}^n \setminus \tilde{\Omega}_m \),

\[
\Pi_k^l (x) \lesssim \prod_{1 \leq i \leq m-1, i \neq j} M f_j^k (x) \sum_l \frac{|Q_m^l|}{|x - x_m^l|^{2n}} \int_{4Q_m^l} |f_j^k (y_j)| \, dy_j
\]
Let\( \eta(x) \) shows that \( \eta \) is bounded from \( L^{\infty} \) to \( L^{\infty} \), bounded from \( L^{p} \) to \( L^{p} \). It is easy to verify that the \( \eta \) claim, \( \eta \) is bounded from \( L^{p} \) to \( L^{p} \), \( \eta \) is bounded from \( L^{p} \) to \( L^{p} \). Therefore, we can now conclude the proof of Lemma 2.2. The assumption (i) tells us that \( T \) is bounded from \( L^{p} \) to \( L^{p} \), \( \eta \) is bounded from \( L^{p} \) to \( L^{p} \). Thus, by our claim, \( T \) is bounded from \( L^{p} \) to \( L^{p} \), \( \eta \) is bounded from \( L^{p} \) to \( L^{p} \). Under the hypothesis of Lemma 2.2, for any \( Q \subset \mathbb{R}^{n} \),

\[
\left( \frac{1}{|Q|} \right) \int_{Q} \left\| f_{i} \right\|_{\infty} dy \leq \prod_{j=1}^{m} \left( \frac{1}{|Q|} \right) \int_{Q} \left| f_{j}(y) \right| dy.
\]

Let \( 1 \leq i \leq m \) and \( l \in \mathbb{N} \), define the operator \( M_{i}^{l} \) by

\[
M_{i}^{l}(f_{1}, \ldots, f_{m}) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \right) \int_{Q} \left| f_{i}(y) \right| dy \prod_{1 \leq j \leq m, j \neq i} \left( \frac{1}{|Q|} \right) \int_{Q} \left| f_{j}(y) \right| dy.
\]

It is easy to verify that

\[
M_{i}(f_{1}, \ldots, f_{m}) \leq \sum_{l=1}^{\infty} 2^{-nl} M_{i}^{l}(f_{1}, \ldots, f_{m})(x).
\]

Let \( M_{w}^{\infty} \) be the weighted centered maximal operator with respect to \( \nu_{w} \), defined as

\[
M_{w}^{\infty} f(x) = \frac{1}{\nu_{\nu_{w}}(B(x, r))} \int_{B(x, r)} |f(y)| \nu_{w}(y) dy.
\]

It is well known that \( M_{w}^{\infty} \) is bounded from \( L^{1}(\mathbb{R}^{n}, \nu_{w}) \) to \( L^{1}(\mathbb{R}^{n}, \nu_{w}) \), and bounded from \( L^{p}(\mathbb{R}^{n}, \nu_{w}) \) to \( L^{p}(\mathbb{R}^{n}, \nu_{w}) \) for \( p \in (1, \infty) \).

**Lemma 2.14.** Let \( p_{1}, \ldots, p_{m} \in [1, \infty) \) and \( p \in (0, \infty) \) with \( 1/p = 1/p_{1} + \cdots + 1/p_{m} \), \( \bar{w} = (w_{1}, \ldots, w_{m}) \in A_{\bar{p}}(\mathbb{R}^{mn}) \). Then there exists a constant \( \theta \in (0, 1) \) such that for any \( l \in \mathbb{N} \) and \( 1 \leq i \leq m \),

\[
M_{i}^{l}(f_{1}, \ldots, f_{m}) \leq C2^{nl} 2^{-\theta l} \prod_{j=1}^{m} \left\{ M_{\nu_{w}} \left[ |f_{j}(y)| w_{j}/\nu_{w} \right] (x) \right\}^{\frac{1}{p}}.
\]
with \( C \) a constant independent of \( i \) and \( l \). Moreover, if \( \min_{1 \leq j \leq m} p_j > 1 \), then there exists a constant \( r > 1 \), such that
\[
\mathcal{M}_l^i(f_1, \ldots, f_m)(x) \lesssim 2^{nl} 2^{d_l} \prod_{j=1}^{m} \left\{ M_{\nu_j} \left[ |f_j|^{p_j} u_j / \nu_j \right]^{r} \right\}^{1/r}.
\]

Lemma 2.14 was essentially given in the proof of Proposition 2.1 in \[1\].

Let \( M^2 \) be the Fefferman-Stein sharp maximal operator, that is,
\[
M^2 f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c| dy.
\]

For \( \delta \in (0, 1) \), let \( M^2_\delta \) be the operator defined by \( M^2_\delta f(x) = \left[ M^2(|f|^{\delta})(x) \right]^{1/\delta} \).

**Lemma 2.15.** Let \( \delta \in (0, 1/m) \). Under the hypothesis of Theorem 1.10, the estimate
\[
M^2_\delta \left( \| (T(f^k_1, \ldots, f^k_m)) \|_{l^r} \right) (x) \lesssim \mathcal{M}_r \left( \| (f^k_1) \|_{l^r}, \ldots, \| (f^k_m) \|_{l^r} \right) (x)
\]
holds true for finite sequences \( \{f_1^k\}, \ldots, \{f_m^k\} \), where and in the following, for \( r \in [1, \infty) \), \( \mathcal{M}_r(h_1, \ldots, h_m)(x) = \mathcal{M}_r(h_1, \ldots, h_m)(x) \); while for \( r \in (0, 1) \),
\[
M_r(h_1, \ldots, h_m)(x) = \left\{ \sum_{j=1}^{\infty} 2^{-nl} \left( \mathcal{M}_r^j(h_1, \ldots, h_m)(x) \right)^r \right\}^{1/r}.
\]

**Proof.** For each fixed \( x \in \mathbb{R}^n \), cube \( Q \) containing \( x \) and \( \{f_1^k\}, \ldots, \{f_m^k\} \), decompose \( f^j_k \) as
\[
f^j_k(y) = f^j_k(y) \chi_{\partial Q} + f^j_k(y) \chi_{\mathbb{R}^n \setminus \partial Q} =: f^k_{j,1}(y) + f^k_{j,2}(y).
\]

Let \( y_Q \in Q \) such that \( \sum_k |T(f^k_1, \ldots, f^k_m)(y_Q)|^r < \infty \). Observing that \( \delta/r < 1 \), we then get
\[
\left\| \left\{ T(f^k_1, \ldots, f^k_m)(y) \right\} \right\|_r^\delta - \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta \leq \left| \sum_k \left| T(f^k_1, \ldots, f^k_m)(y) \right|_r^\delta - \sum_k \left| T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right|_r^\delta \right| \leq \left\| \left\{ T(f^k_1, \ldots, f^k_m)(y) - T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta.
\]

if \( r \in (0, 1] \). On the other hand, it is obvious that
\[
\left\| \left\{ T(f^k_1, \ldots, f^k_m)(y) \right\} \right\|_r^\delta - \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta \leq \left\| \left\{ T(f^k_1, \ldots, f^k_m)(y) - T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta.
\]
holds true when \( r \in (1, \infty) \). Therefore, we can write
\[
\frac{1}{|Q|} \int_Q \left\| \left\{ T(f^k_1, \ldots, f^k_m)(y) \right\} \right\|_r^\delta dy - \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta dy \leq \frac{1}{|Q|} \int_Q \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y) \right\} \right\|_r^\delta dy
\]
\[
+ \frac{1}{|Q|} \sum_l \int_Q \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y) \right\} \right\|_r^\delta dy
\]
\[
+ \frac{1}{|Q|} \int_Q \left\| \left\{ T(f^k_{1,1}, \ldots, f^k_{1,2})(y) - T(f^k_{1,1}, \ldots, f^k_{1,2})(y_Q) \right\} \right\|_r^\delta dy.
\]
where for each term in the sum \( \sum_{i, \ldots, m} \{i_1, \ldots, i_m\} \subset \{1, 2\} \) and at least one \( i_j = 2 \) and one \( i_u = 1 \) with \( 1 \leq j, u \leq m \). We have by Corollary \text{2.13} that
\[
I_1^{1/\delta} \lesssim \prod_{j=1}^{m} \left( \frac{1}{|Q|} \int_{4nQ} \|f_j^k(y_j)\|_{r_j} dy_j \right)
\lesssim \mathcal{M} \left( \|f_1^k\|_{t_1}, \ldots, \|f_m^k\|_{t_m} \right)(x).
\]
(2.17)

For \( \{i_1, \ldots, i_m\} \subset \{1, 2\} \) with at least one \( i_j = 1 \) and \( i_u = 2 \) (\( 1 \leq j, u \leq m \)), set \( \Lambda_{i_1, \ldots, i_m} = \{ j : 1 \leq j \leq m, i_j = 1 \} \). We assume that \( j_0 \in \Lambda_{i_1, \ldots, i_m} \). For each \( y \in Q \), we have by the size condition (1.2) that
\[
|T(f_1^{k_1}, \ldots, f_m^{k_m})(y)| \lesssim \prod_{u \in \Lambda_{i_1, \ldots, i_m}} \int_{8nQ} |f_u^k(y_u)| dy_u
\times \prod_{j \not\in \Lambda_{i_1, \ldots, i_m}} \int_{\mathbb{R}^n \setminus 8nQ} \frac{|f_j^k(y_j)|_{\alpha_m}}{|y - y_j|^{\alpha_m}} dy_j
\lesssim \sum_{l=4}^{\infty} 2^{-nl} \left( \frac{1}{|8nQ|} \int_{8nQ} |f_{j_0}^k(y_{j_0})| dy_{j_0} \right)
\times \prod_{1 \leq j \neq j_0} \left( \frac{1}{|2^l nQ|} \int_{2^l nQ} |f_j^k(y_j)| dy_j \right).
\]

This, along with Hölder’s inequality and Minkowski’s inequality, shows that when \( r \in [1, \infty) \),
\[
\|\{T(f_1^{k_1}, \ldots, f_m^{k_m})\}\|_{l_r} \lesssim \sum_{l=4}^{\infty} 2^{-nlr} \left\{ \sum_k \left( \frac{1}{|8nQ|} \int_{8nQ} |f_{j_0}^k(y_{j_0})| dy_{j_0} \right)^{r_{j_0}} \right\}^{1/r_{j_0}}
\times \prod_{1 \leq j \neq j_0} \left\{ \sum_k \left( \frac{1}{|2^l nQ|} \int_{2^l nQ} |f_j^k(y_j)| dy_j \right)^{r_{j}} \right\}^{1/r_{j}}
\lesssim \mathcal{M}_{j_0} \left( \|\{f_1^k\}\|_{t_1}, \ldots, \|\{f_m^k\}\|_{t_m} \right)(x).
\]

On the other hand, if \( r \in (0, 1] \), we then get from (2.17) and Minkowski’s inequality that
\[
\sum_k |T(f_1^{k_1}, \ldots, f_m^{k_m})(y)|^r
\lesssim \sum_{l=4}^{\infty} 2^{-nlr} \left\{ \sum_k \left( \frac{1}{|8nQ|} \int_{8nQ} |f_{j_0}^k(y_{j_0})| dy_{j_0} \right)^{r_{j_0}} \right\}^{r/r_{j_0}}
\times \prod_{1 \leq j \neq j_0} \left\{ \sum_k \left( \frac{1}{|2^l nQ|} \int_{2^l nQ} |f_j^k(y_j)| dy_j \right)^{r_{j}} \right\}^{r/r_{j}}
\lesssim \sum_{l=4}^{\infty} 2^{-nlr} \mathcal{M}_{j_0}^l \left( \|\{f_1^k\}\|_{t_1}, \ldots, \|\{f_m^k\}\|_{t_m} \right)(x)^r.
\]

Therefore,
\[
\begin{align*}
I_2^{1/\delta} & \lesssim \sum_{i=1}^{\infty} \mathcal{M}_{i, r} \left( \|\{f_1^k\}\|_{t_1}, \ldots, \|\{f_m^k\}\|_{t_m} \right)(x).
\end{align*}
\]
It remains to estimate $I_3$. Note that if $y \in Q$ and $(y_1, \ldots, y_m) \in (\mathbb{R}^n \setminus S_{\ell}(Q))_m$, then $|y - y_q| \leq 2\sqrt{nM}(Q)$ and $4\sqrt{nM}(Q) \leq \min_{1 \leq j \leq m} |x - y_j|$. Thus by (1.6),
\[
\begin{aligned}
&|T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y) - T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y_Q)| \\
&\lesssim \int \mathbb{1}_{(\mathbb{R}^n \setminus S_{\ell}(Q))_m} \{\ell(Q)\}^\gamma \prod_{j=1}^m |f_j^k(y_j)| dy_j \\
&\lesssim \sum_{l=3}^n 2^{-\gamma l} \prod_{j=1}^m \left( \frac{1}{|2^nQ|} \int_{2^nQ} |f_j^k(y_j)| dy_j \right).
\end{aligned}
\]
This, along with Hölder’s inequality and Minkowski’s inequality, implies that
\[
\left\{ \sum_k |T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y) - T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y_Q)|^r \right\}^{1/r} \\
\lesssim \sum_{l=3}^n 2^{-\gamma l} \prod_{j=1}^m \left( \frac{1}{|2^nQ|} \int_{2^nQ} \|f_j^k(y_j)\|_{r'} dy_j \right) \\
\lesssim \mathcal{M}\left(\|f_1^k\|_{r'}^{1}, \ldots, \|f_m^k\|_{r'}^{1}\right)(x)
\]
if $r \in (1, \infty)$, and
\[
\left\{ \sum_k |T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y) - T(f_1^{k_1,2}, \ldots, f_m^{k_m,2})(y_Q)|^r \right\}^{1/r} \\
\lesssim \left\{ \sum_{l=3}^n 2^{-\gamma l} \prod_{j=1}^m \left( \frac{1}{|2^nQ|} \int_{2^nQ} \|f_j^k(y_j)\|_{r'} dy_j \right)^r \right\}^{1/r} \\
\lesssim \mathcal{M}\left(\|f_1^k\|_{r'}^{1}, \ldots, \|f_m^k\|_{r'}^{1}\right)(x)
\]
if $r \in (0, 1)$. Combining the estimates for $I_1$, $I_2$ and $I_3$ leads to (2.16) and then completes the proof of Lemma 2.15.

Proof of Theorem 1.10. Let $p_1, \ldots, p_m \in [1, \infty)$, $p \in (0, \infty)$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{w} = (w_1, \ldots, w_m) \in A_p(\mathbb{R}^{mn})$. We claim that if $r \in (0, 1)$, then for each $i = 1, \ldots, m$,
\begin{align}
(2.18) \quad &\|M_{i,r}(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}. \\
\end{align}
Moreover if $r \in (0, 1)$ and $\min_{1 \leq j \leq m} p_j > 1$, then
\begin{align}
(2.19) \quad &\|M_{i,r}(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|h_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}. \\
\end{align}
In fact, the estimates (2.18) follows from the fact that for some constant $C_0$ depending only on $\theta$ appeared in Lemma 2.14,
\begin{align}
&\{x \in \mathbb{R}^n : M_{i,r}(h_1, \ldots, h_m)(x) > C_0 \lambda\} \\
&\quad \subset \bigcup_{l=1}^\infty \{x \in \mathbb{R}^n : M_{i}(h_1, \ldots, h_m)(x) > 2^{nl}2^{-\theta l/2} \lambda\},
\end{align}
and Lemma 2.14. To prove (2.19), we deduce from Lemma 2.14 that for $r \in (0, 1)$ and $p \in (0, r]$,
\[
\|M_{i,r}(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})}^p \lesssim \sum_{l=1}^\infty 2^{-l\gamma p} \|M_{i}(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})}^p.
\]
On the other hand, for the case of \( r \in (0, 1) \) and \( p \in (r, \infty) \), we have by Minkowski’s inequality that

\[
\|M_{i, r}(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_\omega)}^r \lesssim \sum_{l=1}^{\infty} 2^{-nr} \|M_l(h_1, \ldots, h_m)\|_{L^p(\mathbb{R}^n, \nu_\omega)}^r \lesssim \prod_{j=1}^{m} \|h_j\|_{L^p(\mathbb{R}^n, \nu_\omega)}^r, \text{ if } \min_{1 \leq j \leq m} p_j > 1.
\]

We now prove Theorem 1.10. By a standard limit argument, it suffices to consider the case that \( \{f_1^k\}, \ldots, \{f_m^k\} \) are finite sequences. By Lemma 2.2 and Lemma 2.15, we know that for all \( q_1, \ldots, q_m \in (1, \infty) \) and \( q \in (0, \infty) \) with \( 1/q = 1/q_1 + \cdots + 1/q_m \), \( T \) is bounded from \( L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \) to \( L^{q}(\mathbb{R}^n) \). Again by Lemma 2.2 and Lemma 2.15, we have the estimate

\[
\|M_{\delta}(\|T(f_1^k, \ldots, f_m^k)\|_{L^p(\mathbb{R}^n, \nu_\omega)})\|_{L^p(\mathbb{R}^n, \nu_\omega)} \lesssim M(\|f_1^k\|_{L^p(\mathbb{R}^n, \nu_\omega)}, \ldots, \|f_m^k\|_{L^p(\mathbb{R}^n, \nu_\omega)}) (x)
\]

(2.21)

with \( \delta \in (0, 1/m) \). Let \( p_1, \ldots, p_m \in [1, \infty) \), \( p \in (0, \infty) \) with \( 1/p = 1/p_1 + \cdots + 1/p_m \), and \( w = (w_1, \ldots, w_m) \in A_{\bar{p}}(\mathbb{R}^m) \), By Theorem 3.7 in [16], Proposition 2.1 in [10], and the estimates (2.18) and (2.19), the maximal operators \( M \) and \( M_\delta, q \) are bounded from \( L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m) \) to \( L^{q}(\mathbb{R}^n, \nu_{\bar{w}}) \). Moreover, if \( \max_{1 \leq j \leq m} p_j > 1 \), then these maximal operators are bounded from \( L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m) \) to \( L^{p}(\mathbb{R}^n, \nu_{\bar{w}}) \). Noticing that \( \nu_{\bar{w}} \in A_{p/\delta}(\mathbb{R}^n) \), we then obtain the desired conclusions by (2.21).

3. Proof of Theorem 1.13

Recall that the standard dyadic grid in \( \mathbb{R}^n \) consists of all cubes of the form

\[
2^{-k}([0, 1)^n + j), \quad k \in \mathbb{Z}, \quad j \in \mathbb{Z}^n.
\]

Denote the standard grid by \( D \).

As usual, by a general dyadic grid \( \mathcal{D} \), we mean a collection of cube with the following properties: (i) for any cube \( Q \in \mathcal{D} \), its side length \( \ell(Q) \) is of the form \( 2^k \) for some \( k \in \mathbb{Z} \); (ii) for any cubes \( Q_1, Q_2 \in \mathcal{D}, Q_1 \cap Q_2 \subseteq \{Q_1, Q_2, \emptyset\} \); (iii) for each \( k \in \mathbb{Z} \), the cubes of side length \( 2^k \) form a partition of \( \mathbb{R}^n \).

The following lemma was established in [15].

**Lemma 3.1.** There exists \( 2^n \) dyadic grids \( \mathcal{D}_\alpha \), such that for any cube \( Q \subset \mathbb{R}^n \), there exists a cube \( Q_\alpha \subset \mathcal{D}_\alpha \) which satisfies that \( Q \subset Q_\alpha \) and \( \ell(Q_\alpha) \leq 6\ell(Q) \).

For fixed \( \alpha = 1, \ldots, 2^n \), let \( M^\mathcal{D}_\alpha \) be the maximal operator defined by

\[
M^\mathcal{D}_\alpha(f_1, \ldots, f_m)(x) = \sup_{Q \subset \mathcal{D}_\alpha} \prod_{j=1}^{m} \frac{1}{|Q_j|} \int_Q |f_j(y_j)| dy_j.
\]

Similarly, for \( i, l \in \mathbb{N}, 1 \leq i \leq m \), we define the maximal operator \( M^\mathcal{D}_l(f_1, \ldots, f_m) \) by

\[
M^\mathcal{D}_l(f_1, \ldots, f_m)(x) = \sup_{Q \subset \mathcal{D}_\alpha} \left( \frac{1}{|Q|} \int_Q |f_i(y_j)| dy_j \right) \prod_{1 \leq j \leq m} \left( \frac{1}{|2^l Q|} \int_{2^l Q} |f_i(y_j)| dy_j \right).
\]
It then follows from Lemma 3.1 that
\[ M(f_1, \ldots, f_m)(x) \lesssim \sum_{\alpha=1}^{2^n} M_{\beta}^\alpha(f_1, \ldots, f_m)(x) \]
and
\[ M_1^i(f_1, \ldots, f_m)(x) \lesssim \sum_{\alpha=1}^{2^n} M_{\beta}^{i, \alpha}(f_1, \ldots, f_m)(x). \]

Associated with \( \mathcal{P}_\alpha \), define the sharp maximal function \( M^{2, \mathcal{P}_\alpha} \) as
\[ M^{2, \mathcal{P}_\alpha} f(x) = \sup_{Q \subset \mathcal{P}_\alpha} \inf_{c} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy. \]

As it was proved in [21, p. 153], for \( p \in (0, \infty) \) and \( w \in A_\infty(\mathbb{R}^n) \),
\[ \|f\|_{L^p(\mathbb{R}^n, w)} \lesssim \|M^{2, \mathcal{P}_\alpha} f\|_{L^p(\mathbb{R}^n, w)}, \tag{3.3} \]
provided that \( \|f\|_{L^p(\mathbb{R}^n, w)} < \infty \). Also, repeating the argument in [21, p. 153], we can verify that
\[ \|f\|_{L^{p, \infty}(\mathbb{R}^n, w)} \lesssim \|M^{2, \mathcal{P}_\alpha} f\|_{L^{p, \infty}(\mathbb{R}^n, w)}, \tag{3.4} \]
provided that \( \|f\|_{L^{p, \infty}(\mathbb{R}^n, w)} < \infty \). Let \( M^{2, \mathcal{P}_\alpha}_{\delta} f(x) = \left[ M^{2, \mathcal{P}_\alpha}(f(x))^\delta \right]^{1/\delta} \) with \( \delta \in (0, \infty) \).

The following lemma is a generalization of Lemma 8.1 in [4] in the setting of multi(sub)linear cases, and will play an important role in the proof of Theorem 1.13.

Lemma 3.5. Let \( q_1, \ldots, q_m \in (1, \infty) \), \( q \in (1/m, \infty) \) such that \( 1/q = 1/q_1 + \cdots + 1/q_m \). Then for integer \( 1 \leq i \leq m \), \( \delta \in (0, 1/m) \) and \( \alpha = 1, \ldots, 2^n \),
\[ M^{2, \mathcal{P}_\alpha}_{\delta}(\|\{M^{\mathcal{P}_\alpha}(f_1^k, \ldots, f_m^k)\}\|_{L^q}) \lesssim M\left(\|\{f_1^k\}\|_{L^{q_1}}, \ldots, \|\{f_m^k\}\|_{L^{q_m}}\right) \tag{3.6} \]
and
\[ M^{2, \mathcal{P}_\alpha}_{\delta}\left(\|\{M_i^{\mathcal{P}_\alpha}(f_1^k, \ldots, f_m^k)\}\|_{L^q}\right)(x) \lesssim \ell M_{\delta} M_i\left(\|\{f_1^k\}\|_{L^{q_1}}, \ldots, \|\{f_m^k\}\|_{L^{q_m}}\right)(x) \tag{3.7} \]

Proof. We only prove (3.7). The proof of inequality (3.6) is similar and simpler, and will be omitted. For the sake of simplicity, we only prove (3.7) for \( \mathcal{P}_\alpha = \mathcal{D} \).

Let \( x \in \mathbb{R}^n \) and \( Q_0 \) be a dyadic cube containing \( x \). For each \( y \in Q_0 \) and integer \( v \) with \( 1 \leq v \leq l \), let
\[ A_{\mathcal{D}}^v(y) = \left( \frac{1}{|Q|} \int_{Q} |f_j^k(y, s_j)| \, ds_j \right) \prod_{1 \leq j \leq m} \left( \frac{1}{|2^v Q|} \int_{2^v Q} |f_j^k(y, s_j)| \, ds_j \right), \]
with \( Q \) the unique dyadic cube containing \( y \) and \( \ell(Q) = 2^{-v}(Q_0) \). Also, set
\[ A_{l+1}^v(y) = \sup_{Q \in \mathcal{D}^v} \left( \frac{1}{|Q|} \int_{Q} |f_j^k(y, s_j)| \, ds_j \right) \prod_{1 \leq j \leq m} \left( \frac{1}{|2^v Q|} \int_{2^v Q} |f_j^k(y, s_j)| \, ds_j \right). \]

Observe that for cubes \( Q \subset Q_0 \in \mathcal{D} \), if \( \ell(Q) < 2^{-v}(Q_0) \) for some \( v \in \mathbb{N} \), then \( 2^v Q \subset 2 Q_0 \). Thus,
\[ A_{l+1}^v(y) \lesssim M_{\ell}^i(f_1^k \chi_{2Q_0}, \ldots, f_m^k \chi_{2Q_0})(y). \]
It is easy to verify that
\[ \mathcal{M}_i^{i,D}(f_1^k, \ldots, f_m^k)(y) = \max\{A_i^k(y), \ldots, A_i^k(y), A_{i+1}^k(y), D_i^k(f_1^k, \ldots, f_m^k)\}, \]
with
\[ D_i^k(f_1^k, \ldots, f_m^k) = \sup_{Q \subseteq Q'} \left( \frac{1}{|Q|} \int_Q |f_i^k(y)| \, dy \right) \prod_{1 \leq j < m} \left( \frac{1}{2|Q|} \int_{Q_j} |f_i^k(y)| \, dy \right). \]

Let \( C_0 = \|\{D_i^k(f_1^k, \ldots, f_m^k)\}\|_{L^q}. \) Recall that \( \delta < 1/m < q. \) As in the proof of Lemma 2.15, we can write
\[
\|\{\mathcal{M}_i^{i,D}(f_1^k, \ldots, f_m^k)(y)\}\|_{L^q}^{\delta} - |C_0|^{\delta} \\
\leq \|\{\mathcal{M}_i^{i,D}(f_1^k, \ldots, f_m^k)(y) - D_i^k(f_1^k, \ldots, f_m^k)\}\|_{L^q}^{\delta} \\
\leq \|\{A_i^k(y) + \cdots + A_i^k(y)\}\|_{L^q}^{\delta} + \|\{\mathcal{M}_i^k(f_1^k \chi_{2Q_0}, \ldots, f_m^k \chi_{2Q_0})(y)\}\|_{L^q}^{\delta}.
\]

We now estimate \( \|\{A_i^k(y) + \cdots + A_i^k(y)\}\|_{L^q}. \) For each \( 1 \leq v \leq l \) and \( y \in Q_0, \) applications of Hölder’s inequality and Minkowski’s inequality give us that
\[
\|\{A_i^k(y)\}\|_{L^q} \leq \left( \frac{1}{|Q|} \int_Q \|\{f_i^k(y)\}\|_{L^q} \, dy \right) \\
\times \prod_{1 \leq j < m} \left( \frac{1}{2|Q|} \int_{Q_j} \|\{f_i^k(y)\}\|_{L^q} \, dy \right) \\
\leq \mathcal{M}_i^k\left(\|\{f_1^k\}\|_{L^q}, \ldots, \|\{f_m^k\}\|_{L^q}\right)(y).
\]

This, in turn, implies that
\[
\left( \frac{1}{|Q_0|} \int_{Q_0} \|\{A_i^k(y) + \cdots + A_i^k(y)\}\|_{L^q}^{\delta} \, dy \right)^{1/\delta} \\
\leq \mathcal{M}_s\mathcal{M}_i^k\left(\|\{f_1^k\}\|_{L^q}, \ldots, \|\{f_m^k\}\|_{L^q}\right)(x).
\]

We can now conclude the proof of Lemma 3.5. Recall that
\[
\mathcal{M}_i^k(f_1^k, \ldots, f_m^k)(z) \leq \sum_{j=1}^m Mf_j^k(z).
\]

It is obvious that \( \mathcal{M}_i^k \) is bounded from \( L^1(T^m; \mathbb{R}^n) \times \cdots \times L^1(T^m; \mathbb{R}^n) \) to \( L^{1/m, \infty}(T^q; \mathbb{R}^n) \) with bounded independent of \( i \) and \( l. \) As in the proof of Kolmogorov’s inequality, we deduce that
\[
\left( \frac{1}{|Q_0|} \int_{Q_0} \|\mathcal{M}_i^k(f_1^k \chi_{2Q_0}, \ldots, f_m^k \chi_{2Q_0})(y)\|_{L^q}^t \, dy \right)^{1/t} \\
\leq \prod_{j=1}^m \left( \frac{1}{|2Q_0|} \int_{2Q_0} \|\{f_j^k\}(z)\|_{L^q} \, dz \right) \\
\leq \mathcal{M}\left(\|\{f_1^k\}\|_{L^q}, \ldots, \|\{f_m^k\}\|_{L^q}\right)(x).
\]

Combining the estimates (3.8) and (3.9) then leads to that
\[
\left( \frac{1}{|Q_0|} \int_{Q_0} \|\mathcal{M}_i^{i,D}(f_1^k, \ldots, f_m^k)(y)\|_{L^q}^{\delta} - |C_0|^{\delta} \right)^{1/\delta} \\
\leq \mathcal{M}_s\mathcal{M}_i^k\left(\|\{f_1^k\}\|_{L^q}, \ldots, \|\{f_m^k\}\|_{L^q}\right)(x) + \mathcal{M}\left(\|\{f_1^k\}\|_{L^q}, \ldots, \|\{f_m^k\}\|_{L^q}\right)(x),
\]
and leads to the desired conclusion for $\mathcal{M}_i^{L,D}$.

Proof of Theorem 1.13. We only prove the conclusion for $\mathcal{M}_i$ ($1 \leq i \leq m$).

Observe that for $\{f_1^k, \ldots, f_m^k\}$ finite sequences. We first consider the case $p_1, \ldots, p_m \in (1, \infty)$. Let $q_1, \ldots, q_m \in (1, \infty)$, $\vec{w} = (w_1, \ldots, w_m) \in A_p(\mathbb{R}^{nm})$, $\alpha = 1, \ldots, 2^n$ and $\delta \in (0, 1/m)$, we obtain from (3.3), Lemma 3.5 and Lemma 2.14, that

$$\|\{\mathcal{M}_i^{L,D}(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim\|\mathcal{M}_i^{L,D}(\{f_1^k, \ldots, f_m^k\})\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})}$$

$$\lesssim \|M_i(\{f_1^k\})_{l_1}, \ldots, \|f_m^k\|_{l_{nm}}\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})}$$

$$+ \|M_i(\{f_1^k\})_{l_1}, \ldots, \|f_m^k\|_{l_{nm}}\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim 2^{nq_{\delta}} 2^{-\delta q_1} \prod_{j=1}^{m} \|f_j^k\|_{L^{\gamma_j}((\vec{w}; \mathbb{R}^n, w_j))},$$

since $\nu_{\vec{w}} \in A_p(\mathbb{R}^n)$. This, via (3.2), yields

$$(3.10) \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \|f_1^k\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j^k\|_{L^{\gamma_j}((\vec{w}; \mathbb{R}^n, w_j))},$$

Observe that for $q \in (1, \infty)$,

$$(3.11) \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)(x)\}\|_{\ell^q} \lesssim \sum_{i=1}^{\infty} 2^{-ln\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)(x)\}\|_{\ell^q},$$

and for $q \in (0, 1]$,

$$\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)(x)\}\|_{\ell^q} \lesssim \sum_{i=1}^{\infty} 2^{-nq_{\delta}} \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)(x)\}\|_{\ell^q}.$$

Therefore,

$$\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \sum_{i=1}^{\infty} 2^{-ni} \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})},$$

when $p \in (1, \infty)$; and

$$\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \sum_{i=1}^{\infty} 2^{-nq_{\delta}} \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})},$$

when $q \in (1, \infty)$ and $p \in (0, 1]$ or $q \in (0, 1]$ and $p \in (0, q]$; and

$$\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \sum_{i=1}^{\infty} 2^{-nq_{\delta}} \|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})},$$

when $q \in (0, 1]$ and $p \in (q, \infty)$. We now deduce from (3.10) that

$$\|\{\mathcal{M}_i(f_1^k, \ldots, f_m^k)\}\|_{L^p(\vec{w}; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j^k\|_{L^{\gamma_j}((\vec{w}; \mathbb{R}^n, w_j))}.$$
which, together with (3.2) gives us that

\[
(3.13) \quad \left\| \{ M_i^j(f_{k_1}, \ldots, f_{k_m}) \} \right\|_{L^p, \infty} \leq 2^{nl} - \frac{\theta l}{2} \prod_{j=1}^{m} \left\| f_j^k \right\|_{L^p_j (\mathbb{R}^n, w_j)}.
\]

On the other hand, as in the inequality (2.20), we get from (3.11), (3.12) and (3.13) that

\[
\nu_{\mathbb{R}^n} \left( \{ x \in \mathbb{R}^n : \left\| \{ M_i^j(f_{k_1}, \ldots, f_{k_m}) \} \right\|_{l_q} > C \theta \lambda \} \right)
\]

\[
\lesssim \sum_{l=1}^{\infty} \nu_{\mathbb{R}^n} \left( \{ x \in \mathbb{R}^n : \left\| \{ M_i^j(f_{k_1}, \ldots, f_{k_m}) \} \right\|_{l_q} > 2^{nl} - \frac{\theta l}{2} \lambda \} \right)
\]

\[
\lesssim \lambda^{-p} \prod_{j=1}^{m} \left\| f_j^k \right\|_{L^p_j (\mathbb{R}^n, w_j)}.
\]

This completes the proof of Theorem 1.13.

\[\Box\]

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