Two applications of the gauge/gravity duality

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ACADEMIC DISSERTATION

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Abstract

The anti de Sitter/conformal field theory correspondence, or AdS/CFT for short, is an equivalence between a string theory with gravity defined on a space and a quantum field theory without gravity on the boundary of this space. The correspondence makes it possible to compute observables in strongly interacting field theories. We will focus on the weak form of correspondence, which explains how classical supergravity in five dimensions is related to four dimensional strongly interacting field theory. This thesis is divided into two parts. The first part introduces the correspondence, and the second part contains three original research publications. The publications present two different applications of the correspondence.

The first application is to quantum chromodynamics. In particular we derive results on the spatial string tension at finite temperature and thermodynamics of quantum chromodynamics.

The second application is to condensed matter physics, namely quantum Hall transitions. We derive results for frequency independent and dependent conductivities in quantum Hall transitions, and find a universal behavior in our solutions.
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List of Publications

The content of this thesis is based on the following research articles:


The Author’s Contribution to the Joint Publications

I. The present author did analytical calculations, and was responsible for the numerical solutions and plots. Participated in writing the paper.

II. The present author was responsible for some of the analytic calculations, and numerical solutions and plots.

III. The present author did analytical calculations, in particular, for the conductivity, and was responsible for numerical solutions and plots.
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Chapter 1

Introduction

Maldacena conjectured in 1998 that string theory in a five dimensional anti de Sitter space is equivalent, or in other words, dual, to a supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in one four spacetime dimension [1]. Since then, there have been hundreds of tests of the conjecture finding evidence for it. Maldacena’s duality is called the AdS/CFT correspondence. This thesis investigates two applications of the correspondence.

The correspondence is an example of the holographic principle [2, 3]. The principle states that the gravitation degrees of freedom and their dynamics in a volume of space can be encoded on a boundary to the region. In popular writings about the principle, Bekenstein has even wondered if all the information in the Universe could be written “in a grain of sand”.

The duality has been applied in several different cases. Perhaps the most investigated one is the attempt to understand nonperturbative quantum chromodynamics (QCD) better. Often called AdS/QCD, the program has been motivated by the long pursuit of solving QCD with traditional methods. The problems with QCD are in the low energy regime, where the theory becomes strongly interacting, and thus nonperturbative. The correspondence gave hope to solving these problems because it is well suited to strong coupling physics. There has been a lot of advance and new insights but an exact QCD dual is yet to be found. Some reviews of AdS/QCD are [4, 5, 6].

In addition to QCD, the duality has been applied to condensed matter physics as well. In condensed matter physics many of the theories are strongly interacting, like QCD at low energies. In comparison to experiments in the Large Hadron Collider and high energy physics, many condensed matter experiments are easy to perform. Also in contrast to experiments at high energy physics, in condensed matter physics there are many different models to solve, and experiment with. This topic is reviewed in [7, 8].

This thesis is divided into an introductory part and three articles. The introductory part is meant to be an overview of the ideas behind the three papers.

The second chapter introduces the AdS/CFT correspondence. We will describe the basic components, anti de Sitter spaces and conformal field theories and use a scalar field as an illustration of the duality. We will not go through the technical details of deriving the duality.

The third chapter introduces concepts for thermal field theories in the duality. We will then present the model by Kiritsis et al. [9], which was the idea we based the two first papers in this thesis on. We will also shortly review results from our papers.

The fourth chapter introduces a finite charge density and focuses on condensed matter
applications. We will also present a method to solve conductivity in the boundary theory using the duality.

The three articles are the main part of this thesis. In the first article we compute the spatial string tension of finite temperature QCD in a gauge/gravity model. In the second article we study a gauge/gravity model for thermodynamics of a gauge theory with a single parameter, the running coupling. These models are based on relating the scale in the fifth dimension and the renormalization scale. This also gives us a way to calculate the scalar field potential needed in our model. The third article computes electrical transport at quantum Hall critical points in a gauge/gravity dual model. We compare our results with field theory computations, and find good agreement.
Chapter 2

AdS/CFT duality

2.1 Anti de Sitter spaces

To start investigating the duality we first need to familiarize ourselves with anti de Sitter spaces (AdS) and conformal field theory (CFT). We begin by exploring the former first.

De Sitter (dS) and anti de Sitter spaces are solutions of the Einstein-Hilbert action with a cosmological constant term. The action is

$$S = \frac{1}{16\pi G_d} \int d^d x \sqrt{|g|} (R + \Lambda). \quad (2.1)$$

Here $G_d$ is Newton’s gravitational constant in $d$ dimensions, $g$ is the determinant of the metric, $R$ is the Ricci scalar, and $\Lambda$ is the cosmological constant. The action is given in the Minkowski signature with a mostly plus convention. For the Euclidean signature the sign of the action would be negative. The solutions to the vacuum equations of motion are derived from the action (2.1) by the action principle as follows:

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{1}{2} \Lambda g_{\mu \nu} \quad \Rightarrow \quad (2.2)$$

$$R = \frac{d}{2 - d} \Lambda \quad \Rightarrow \quad (2.3)$$

$$R_{\mu \nu} = \frac{\Lambda}{2 - d} g_{\mu \nu}.$$ 

Depending on the sign of the cosmological constant the solution is either anti de Sitter or de Sitter. In our notation anti de Sitter corresponds to a positive cosmological constant and de Sitter to negative. These spaces are also called Einstein spaces, since their Ricci tensor is proportional to the metric tensor. The Riemann tensor components turn out to be

$$R_{\mu \nu \rho \sigma} = \frac{R}{d(d-1)} (g_{\nu \sigma} g_{\mu \rho} - g_{\nu \rho} g_{\mu \sigma}). \quad (2.4)$$

These solutions are called maximally symmetric because they have $d(d+1)/2$ Killing vectors or isometries, meaning that all the coordinates are mathematically equivalent and no direction is preferred. Examples of such spaces are $d$-dimensional spheres, flat spaces, and the already mentioned AdS and dS spaces.
From the maximal symmetry also follows from equation (2.3) that the Ricci tensor is constant, so that the curvature is everywhere constant. For \( d > 2 \) the curvature in anti de Sitter is negative and thus the space curves hyperbolically, for de Sitter spaces the curvature would be parabolic, and the sphere stands between these two types of solutions.

### 2.1.1 Anti de Sitter embedded in \( \mathbb{R}^n \)

Consider an \((n+1)\)-dimensional AdS\(_{n+1}\) as a submanifold of an \((n+2)\)-dimensional embedding space with coordinates and metric,

\[
(y^a) = (y^0, y^1, ..., y^{n+1})
\]

\[
\eta_{ab} = (1, -1, -1, ..., -1, 1).
\]

The scalar product of two vectors, given by

\[
x \cdot y = \eta_{ab} x^a y^b
\]

is conserved by the group \( \text{SO}(2, n) \) acting like

\[
y^a \rightarrow y'^a = \Lambda^a_b y^b, \quad \Lambda^a_b \in \text{SO}(2, n).
\]

One way to define AdS\(_{n+1}\) is the set of points which satisfy

\[
y^2 = L^2,
\]

for some constant \( L \). If, instead of (2.7), we had \( y^2 = -L^2 \), the space would be different and called de Sitter space.

The definition given above also confirms that the space is maximally symmetric. AdS\(_{n+1}\) is clearly \( \text{SO}(2, n) \) invariant and its dimension is \((n+1)(n+2)/2\) just as expected from an \((n+1)\)-dimensional space.

We can also investigate the space in stereographic coordinates \( x^\mu \) such that

\[
y^0 = L \frac{1 + x^2}{1 - x^2},
\]

\[
y^\mu = L \frac{2x^\mu}{1 - x^2}, \quad \mu = 1, ..., n+1
\]

where now

\[
x^2 = (x^1)^2 + ... (x^n)^2 - (x^{n+1})^2.
\]

This set of coordinates \((L, x^\mu)\) satisfies \( y^2 = L^2 \) and we can work out the metric to be

\[
ds^2 = dL^2 - \frac{4L^2}{(1 - x^2)^2} dx^2.
\]

Here we can separate the radial part from the AdS part and note that the latter is conformally flat

\[
g_{\mu\nu} = \frac{4L^2}{(1 - x^2)^2} \eta_{\mu\nu}.
\]
We can prove this satisfies equations (2.1) and (2.3) with
\[ \Lambda = \frac{n(n-1)}{L^2}. \] (2.11)

Anti de Sitter metric can be written in many coordinates. Let us further change our coordinates to
\[ u = y^0 + iy^{n+1}, \quad v = y^0 - iy^{n+1}, \] (2.12)
so the equation (2.7) becomes
\[ y^2 = uv - \sum_{i=1}^{n} (y^i)^2 = L^2. \] (2.13)

This is equivalent to investigating the Euclidean AdS\(_{n+1}\). The metric can be further transformed into the form used by Maldacena [1]. Defining, for positive \( u \),
\[ x^\alpha = \frac{y^\alpha}{u}, \quad \alpha = 1, \ldots, n \]
\[ x^2 = \sum_{\alpha=1}^{n} (x^\alpha)^2 \]
we get eventually the metric
\[ ds^2 = L \left( \frac{du^2}{u^2} + u^2 dx^2 \right). \] (2.14)

This show us that the metric has slices (\( u = \text{constant} \)) which are conformal to four-dimensional Minkowski space-time. This is the reason these are called Poincaré coordinates. The coordinate \( u \) runs from zero to infinity. The warp factor \( u^2 \) multiplies the Minkowski metric, which means that an observer on a certain Minkowski slice sees all lengths rescaled by a factor \( u \).

In these coordinates, the \( u = \infty \) is referred to as the boundary of the space and \( u = 0 \) is a horizon. If the whole AdS\(_5\) is viewed as a hyperboloid, this patch only covers half of the space. Another coordinate system, global coordinates, covers the full space.

Yet one more transformation is \( z = L/u \) and we get \( du/u = -dz/z \) for the coordinates used by Witten [10]
\[ ds^2 = \frac{L^2}{z^2} (dz^2 + dx^2). \] (2.15)

Here the boundary is now at \( z = 0 \) and the horizon at \( z = \infty \).

### 2.1.2 Black holes

Black holes are interesting objects. They combine extremal energy densities and quantum mechanics with thermodynamics. First we will inspect Schwarzschild black holes. They are solutions to vacuum Einstein equations (2.1) with \( \Lambda = 0 \). The solutions are also assumed to be spherically symmetric and static
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \] (2.16)
In four dimensions there are four different black hole solutions. These are either rotating or nonrotating and charged or uncharged. If we look for a solution describing a nonrotating and uncharged mass we find the Schwarzschild solution

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),$$

(2.17)

where $c$ is the speed of light, given here for reference, $\tau$ is the proper time measured by a moving clock, $t$ is the time coordinate measured at infinity, $r, \theta$ and $\varphi$ are the coordinates related to radius and angle, and $r_s = 2GM/c^2$ is the Schwarzschild radius. Comparing $r_s$ with some common distances and masses we note that the ratio $r_s/r$ is very small for well-known objects such as the Earth or the Sun. This illustrates the extreme conditions prevailing in black holes.

The metric has two singularities, at $r = r_s$ and $r = 0$. The first one is a coordinate singularity, since even if the metric component blows up, the curvature invariants are finite. This means that we can transform the metric into a form that does not contain any singular components at $r_s$, for example Eddington-Finkelstein or Kruskal-Szekeres coordinates to name a few. The latter at $r = 0$ is a true physical singularity. At the singularity the mathematical model of the spacetime breaks down.

The first singularity at the Schwarzschild radius $r = r_s$ is a coordinate singularity and it coincides with the event horizon of the black hole. For all values of $r < r_s$ all future-directed paths inevitably lead to decreasing $r$. There is no problem however, for an object outside the event horizon, $r > r_s$ to end up in it, but once it enters the horizon, it will never be able to return. All lightlike paths inside the horizon always end up falling ever deeper into the hole.

Quantum thermodynamics of black holes

The second law of thermodynamics states that in a closed system entropy never decreases. We know that black holes have strong gravitational fields and they draw in all matter and energy within the Schwarzschild radius. To an outside observer, matter falling into a black hole carries entropy along with it which seems to be lost forever behind the horizon. This leads to an apparent violation of the second law of thermodynamics, unless the black holes itself carries entropy, with a corresponding increase compensating the loss of matter entropy. This idea motivated Bekenstein to suggest in his PhD thesis that black holes have an entropy associated with its surface area.

Hawking confirmed Bekenstein’s conjecture [13, 14] that the entropy of a $d$-dimensional black hole is proportional to the area of the event horizon

$$S_{BH} = \frac{A}{4G_d},$$

(2.18)

In gravity, this entropy, for fixed energy, is maximal in a given volume. Thus every region in space has its maximal entropy scaling with the area of the boundary, not of the volume. The holographic principle then states that the maximum entropy in a region can be related to the number of degrees of freedom of a local quantum field theory in one less dimension [2, 3]. The AdS/CFT correspondence is a realization of this principle.
2.2 Conformal field theory

If a theory has no scales or dimensionful parameters, it is classically scale invariant. One of the simplest examples of such a theory is the scalar field with the following action

\[ S = \int dx^4 \left( \frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{4!} \phi^4 \right). \]  

This action is invariant under simultaneous space-time coordinate rescaling, \( x \rightarrow x/\lambda \), and field rescaling

\[ \phi(x) \rightarrow \lambda^\Delta \phi(x), \]  

where the weight \( \Delta \) is called the scaling dimension of the field \( \phi \). In the case of quartic interactions it is the same as the canonical mass dimension. In this example it was crucial to not have a mass term as it would spoil the scale invariance.

We must also note that this theory will not be scale invariant after we take into account quantum corrections. It is not simple to know whether or not the theory will be scale invariant after the quantum corrections. The AdS/CFT correspondence involves theories which are exactly scale invariant even after quantum corrections.

2.2.1 The conformal group

The invariance under scale transformations such as (2.20) often implies invariance under a larger group of transformations known as conformal transformations.

In \( d \) dimensions a conformal transformation is the rescaling of the line element by a change of coordinates,

\[ x_\mu \rightarrow x'_\mu \implies dx^2 = dx'^2 = \Omega^2(x) dx^2, \]  

where \( \Omega(x) \) is an arbitrary function of the coordinates. These transformations rescale lengths but preserve all angles. Clearly, scale transformations are a part of conformal transformations.

To find the others, we take the infinitesimal version \( x'_\mu = x_\mu + u_\mu, \Omega(x) = 1 + \omega(x)/2 \) to obtain

\[ \partial_\mu u_\nu + \partial_\nu u_\mu = \omega(x) \eta_{\mu\nu}. \]  

From the above equation we can take a trace and get \( d\omega = 2\partial_\mu u^\mu \). Substituting this back to the above equation, we find an equation describing conformal transformations at the infinitesimal level

\[ \partial_\mu u_\nu + \partial_\nu u_\mu - \frac{2\partial_\alpha u^\alpha}{d} = 0. \]  

In two dimensions there are infinitely many solutions, but in higher dimensions, the solutions are more restricted [15]. Taken all together we find the following continuous transformations

- Translations \( a_\mu \), generated by \( P_\mu \).
- Lorentz rotations \( \omega_{\mu\nu} x_\nu \) (\( \omega_{\mu\nu} = -\omega_{\nu\mu} \)), generated by \( M_{\mu\nu} \).
- Dilations \( \lambda x_\mu \), generated by \( D \), and
- Special conformal transformations \( b_\mu x^2 - 2x_\mu b_\nu x^\nu \), generated by \( K_\mu \).
We can now calculate that for $d$-dimensions we have

$$d + \frac{d(d - 1)}{2} + 1 + d = \frac{(d + 1)(d + 2)}{2}$$

(2.24)

This number is the same as the dimension of $SO(2, d)$, which is not a coincidence, as the conformal group is isomorphic to it when $d$ is larger than 2.

We can now see that under rather mild conditions, scale invariance implies conformal invariance. Let us construct a Noether current associated with the conformal transformations $\delta x^\nu$,

$$J_\mu = T_{\mu\nu} \delta x^\nu.$$  

(2.25)

For the current corresponding to translations to be conserved, we need $\partial^\mu T_{\mu\nu} = 0$, and for Lorentz transformations, we need $T_{\mu\nu}$ to be symmetric. The condition for the scale invariance is the tracelessness of the stress energy tensor. This can be seen from the condition for the conservation of the dilation current

$$\partial^\mu (T_{\mu\nu} x^\nu) = T_{\mu}^\mu = 0.$$  

(2.26)

We can now conclude that in a scale and Poincaré invariant theory the conformal currents are always conserved

$$\partial^\mu (T_{\mu\nu} u^\nu) = \frac{1}{d} \partial^\alpha u_\alpha T_\mu^\mu = 0.$$  

(2.27)

Scale invariance even without Poincare invariance often implies conformal invariance, but not always. Discussion of scale invariant but non-conformal theories can be found in [16, 17].

### 2.2.2 Conformal quantum field theories

In a quantum theory, scale invariance is broken by a renormalization scale $\mu$. The scale symmetry is then said to be anomalous. The couplings $g(\mu)$ are generally running and this is governed by the equation

$$\mu \frac{dg(\mu)}{d\mu} = \beta(g),$$  

(2.28)

where $\beta(g)$ is the beta function.

Because quantum field theory cannot depend on the renormalization scale $\mu$, one finds the Callan-Symanzik equations for the evolution of the $n$-point functions. For discussion about dilatations and Ward identities in this context, see [18].

Let us take as an example the pure Yang-Mills theory

$$S = \int d^4x \frac{1}{4g^2} \text{Tr}(F_{\mu\nu} F_{\mu\nu}).$$  

(2.29)

Even though this theory is classically scale invariant, dimensional transmutation introduces a dimensionful parameter and the stress energy tensor is no longer traceless,

$$T_\mu^\mu \propto \beta(g) F_{\mu\nu} F_{\mu\nu}. $$  

(2.30)
Figure 2.1: Beta function, which has two fixed points, $g = 0$ and $g = g^\star$. The arrow depicts the way the equation (2.28) flows.

This is also known as the trace anomaly or interaction measure. Due to the same mechanism the canonical dimension of the field is corrected by an anomalous dimension

$$\Delta = d + \gamma (g), \quad \gamma = \frac{1}{2} \mu \frac{d \ln Z}{d \mu}. \quad (2.31)$$

A quantum field theory can be conformally invariant in two ways

- At the fixed point, $g^\star$, of the renormalization group. These are the points where $\beta (g^\star) = 0$ and the stress energy tensor becomes traceless. This is illustrated in figure 2.1.

- The theory might have $\beta (g) = 0$ for all $g$. A relevant example for this thesis is the $\mathcal{N} = 4$ super Yang-Mills [19]. It is a theory with non-abelian gauge fields, four Weyl fermions and six real scalars. The number of components each add up to exactly zero beta function at least up to the third loop, possibly all loops [20].

The investigation of the behavior of the beta function in a specific case was of key importance in our papers [21, 22]. We modified the beta function slightly and used it as an input. We were then able to calculate spatial string tension and compare to results in the literature. The methods are described in section 3.2.

2.3 AdS/CFT correspondence

The AdS/CFT correspondence is a particular realization of the holographic principle: it establishes an equivalence between a higher dimensional quantum gravity theory in an asymptotically anti de Sitter space with a conformal quantum field theory. In more detail [1, 10, 23] it connects an SU($N_c$) Super Yang-Mills with four supersymmetries, that is $\mathcal{N} = 4$ SYM, in four dimensions and the type IIB superstring theory in $\text{AdS}_5 \times S^5$. The correspondence relates a scale invariant theory $\mathcal{N} = 4$ SYM with a Yang-Mills coupling, $g_{YM}$, and the number of colors, $N_c$, with string theory characterized by the string coupling, $g_s$, and the string length, $l_s$. These parameters are connected via the relations [19, 24, 25]

$$g_s = g_{YM}^2, \quad L^4 = 4\pi g_{YM}^2 N (\alpha')^2,$$
where $L$ is the radius of both AdS$_5$ and $S^5$ spaces and $\alpha' = l_s^4$, with $l_s$ being the string length. This form of the correspondence is called the strong form.

2.3.1 The 't Hooft and large $\lambda$ limit

For the coupling constant we can consider two limits. One is keeping the 't Hooft coupling

$$\lambda = g_{YM}^2 N = g_s N \quad (2.32)$$

fixed and letting $N \to \infty$. In the Yang-Mills side, this limit is well-defined in perturbation theory. It corresponds to a topological expansion of the Feynman diagrams of the field theory. On the AdS side, this limit can be interpreted as the weak coupling string perturbation theory, this can be seen when the string coupling is re-expressed in terms of the 't Hooft coupling $g_s = \lambda/N$. Since $\lambda$ is fixed, the 't Hooft limit corresponds to weak coupling as $N$ is taken to infinity.

After the 't Hooft limit has been taken the only free parameter is $\lambda$. The perturbation theory of the quantum field corresponds to $\lambda \ll 1$. On the AdS side, however, it is easier to take $\lambda \gg 1$. The latter fact can be seen from analyzing the effective string action as an expansion in $\alpha'$ is

$$\mathcal{L} = a\alpha' R + b(\alpha')^2 R^2 + c(\alpha')^3 R^3 + \ldots \quad (2.33)$$

where $R$ is the Riemann tensor. The scale of the Riemann tensor is set by the AdS radius $L$,

$$R \sim \frac{1}{L^2} = \frac{\lambda^{-1/2}}{\alpha'} \quad (2.34)$$

so the expansion of the effective action becomes qualitatively an expansion in power of $\lambda^{-1/2}$,

$$\mathcal{L} = a\lambda^{-1/2} + b\lambda^{-1} + c\lambda^{-3/2} + \ldots \quad (2.35)$$

These limits make the original statement by Maldacena weaker but at the same time provide us a tool to calculate many quantities. Without these limits we would have to deal with a quantum theory of gravity which is difficult.

2.3.2 The duality

The AdS/CFT duality can be simply stated as the equivalence of $(d+1)$-dimensional classical gravity theory on AdS$_{d+1}$ vacuum to the large $N$ limit of a strongly coupled $d$-dimensional CFT in flat space. This can be formulated in $d = 4$ as

$$Z_{\text{CFT}}[\phi_0] = \int D\mathcal{O} e^{i\mathcal{S}_{\text{CFT}}[\mathcal{O}]+i \int d^4x \phi_0 \mathcal{O}(x)} = Z_{\text{Sugra}}[\phi(x,z \to 0) = \phi_0(x)], \quad (2.36)$$

where, on the left hand side, $\mathcal{O}(x)$ and $\phi_0(x)$ are, respectively, operators and sources of the CFT. On the right hand side we have the partition function evaluated at the classical solution of the supergravity equations of motion, with the boundary condition $\lim_{z \to 0} z^\Delta x^d \phi(x,z) = \ldots$.
The duality says that to any scalar there corresponds a CFT scalar operator, to a gauge field a current, and to the metric a stress-energy tensor in CFT
\[
\phi \leftrightarrow O, \\
A_\mu \leftrightarrow J_\mu, \\
g_{\mu\nu} \leftrightarrow T_{\mu\nu}.
\]
With (2.36) we can calculate the on-shell actions on the right hand side and then find correlation functions of the operators on the left hand side. To find the correct values for the expectation values of the operators we need to renormalize the expressions carefully. This has been done in [26, 27].

We can also calculate correlators on the right hand side with the action perturbed by the source \(\phi_0\). The source can be extended to the bulk, \(\phi_0(x) \to \phi(x,z)\) with the extra coordinate \(z\). With suitable boundary conditions we can fix \(\phi(x,z)\) completely and as a result we have a mapping between the bulk field and the boundary field [10, 23].

### 2.3.2.1 Scalar field

Let us see how the field-operator correspondence tells us the way the conformal dimension of an operator is related to bulk field properties. We will study a massive scalar field \(\phi\) coupled to some scalar gauge invariant operator \(O\) [10, 29]. Take the metric in Poincaré coordinates (2.15)
\[
ds^2 = L^2 \frac{dz^2 + dx^a dx^b}{z^2} \equiv g_{ab} dx^a dx^b, \quad a = 0, \ldots, d. \tag{2.37}
\]
The bulk classical action is
\[
S = -\frac{k}{2} \int d^{d+1}x \sqrt{g} (g^{ab} \partial_a \phi \partial_b \phi + m^2 \phi^2), \tag{2.38}
\]
where \(k\) is a normalization constant. For the given metric \(\sqrt{g} = (L/z)^{d+1}\), and the equation of motion in Fourier space reads
\[
(z^2 k^2 - z^{d+1} \partial_z (z^{1-d} \partial_z) + m^2 L^2) \phi_k(z) = 0. \tag{2.39}
\]
The solutions of this equation are Bessel functions. The solutions near the boundary can be studied by using \(\phi_k(z) = z^\Delta\) in (2.39):
\[
(z^2 k^2 - \Delta(\Delta - d) + m^2 L^2) z^\Delta = 0, \tag{2.40}
\]
which yields
\[
\Delta(\Delta - d) = m^2 L^2 \tag{2.41}
\]
in the \(z \to 0\) limit. This has solutions
\[
\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}. \tag{2.42}
\]
Now we can write a general solution to the equation (set \(\Delta = \Delta_+\) and \(\Delta_- = d - \Delta\)) and Fourier transform it back to coordinate space
\[
\phi(x,z) \approx C_1(x)(z^{d-\Delta} + \ldots) + C_2(x)(z^\Delta + \ldots) \quad \text{as} \ z \to 0. \tag{2.43}
\]

Note that
The solution proportional to $z^{\Delta_{-}}$ is always larger near the boundary,

- $\Delta_{+} > 0$, therefore $z^{\Delta_{+}}$ always decays near the boundary.

We must set some boundary conditions for the field. To have a finite source we can define the boundary conditions as $[10, 23]$

$$\lim_{z \to 0} z^{\Delta_{-}} \phi(x, z) = \phi_{0}(x),$$

which can be compared with (2.43) to identify

$$C_{1}(x) = \phi_{0}(x).$$

Similarly we also define

$$\lim_{z \to 0} z^{-\Delta} \phi(x, z) = \phi_{1}(x).$$

We can also compute expectation values for the operators. They are calculated with the help of the equation (2.36). From that equation we see that

$$\langle \mathcal{O} \rangle = -i \frac{\delta Z[\phi_{0}]}{\delta \phi_{0}} = \frac{\delta S[\phi_{0}]}{\delta \phi_{0}},$$

where the limit $N \to \infty$ has been taken, and $Z = e^{iS}$. Just like for one dimensional dynamics governed by

$$S[x] = \int_{t_{i}}^{t_{f}} dt L(x, \dot{x})$$

the variation of the action with respect to the initial value of the coordinates is the canonical momentum

$$\frac{\delta S}{\delta x(t_{i})} = \Pi(t_{i}).$$

We can think of the radial coordinate of $AdS$ as time and take into account possible boundary terms in the action and find

$$\frac{\delta S[\phi_{0}]}{\delta \phi_{0}} = \lim_{z \to 0} \left( - \frac{\delta S[\phi_{0}]}{\delta \partial_{z} \phi_{0}} + \frac{\delta S_{\partial z}[\phi_{0}]}{\delta \phi_{0}} \right) = \lim_{z \to 0} \Pi[\phi].$$

The boundary terms differ for different actions and a detailed discussion can be found in $[26, 27]$.

In the case of scalar function, the boundary term in the action is

$$S_{bdry} = -\frac{k}{2} \int_{\partial} d^{d}x \sqrt{g} g^{zz} \phi \partial_{z} \phi.$$
In general the on-shell action has to be renormalized because it often has infrared divergences because of the integration region near the boundary of AdS, just as we did in our example. These infrared divergences are dual to ultraviolet divergences in the gauge theory. The whole method of removing these divergences is well understood and it is called the holographic renormalization.

After calculating the boundary terms, we can identify the expectation value of the operator corresponding to the scalar previously discussed, also in [7], as

\[ \langle O \rangle = (2\Delta - d)\phi_1(x). \]  

This is often interpreted as the normalizable \( \phi_1 \) mode giving the expectation value and non-normalizable mode \( \phi_0 \) being the source. The formula is very important for applications and it applies also in the real-time case.

We will next briefly describe linear response. In the linear response theory

\[ \delta \langle O \rangle = \delta \phi_0 G + O((\delta \phi_0)^2). \]  

Equation (2.52) then implies

\[ G(\omega, k) = \frac{\delta \langle O \rangle}{\delta \phi_0} = (2\Delta - d)\frac{\phi_1(\omega, k)}{\phi_0(\omega, k)}. \]  

We will need this result when discussing applications in chapters 3 and 4.

A noteworthy fact is that in AdS spaces the exponents in (2.43) are real, if

\[ m^2 L^2 > -\frac{d^2}{4}. \]  

This tells us that negative values for the square of the mass are allowed in the curved spacetime as opposed to flat spacetime with strict restrictions on the positiveness of the mass. The equation relating the mass and dimension is called the Breitenlohner-Freedman bound [30].

In the next chapters we will discuss specific applications of the duality. The first application is to quantum chromodynamics (QCD) and we will study its thermodynamics in some detail. The other application is to condensed matter physics in a simple setting. The applications are related to the publications by the author in this thesis.
Chapter 3

Application to particle physics

As we have learnt, the AdS/CFT correspondence is a useful tool for studying strongly interacting field theories. New methods, also known as gauge/gravity dualities, have an even broader area of applicability. These methods have been able to establish geometrical pictures of some non-conformal theories. One of the first extensions of the correspondence was on quantum field theories at finite temperature, and confinement [31].

One of the most famous results calculated using the gauge/gravity duality is the ratio of shear viscosity to entropy density [32, 33]

\[ \frac{\eta}{s} = \frac{\hbar}{4\pi}. \]  \hspace{1cm} (3.1)

This value is almost identical to the numerical result obtained from experimental data [34]. The model that was used to derive the result (3.1) was \( \mathcal{N} = 4 \) SYM at finite temperature. The result is reviewed in [35].

The duality has been applied to several different theories in both particle and condensed matter physics. This chapter will present an application to QCD. There are, of course, other approaches to solving QCD using dualities, for a few examples, see papers about Sakai-Sugimoto model and application to quark gluon plasma in heavy ion collisions [36, 37].

3.1 Finite temperature

In the path integral formulation, we can write the partition function of a quantum field theory at finite temperature \( T \) by choosing periodic boundary conditions for closed Euclidean time paths of length \( \beta = 1/T \) [38, 39]. For a scalar field \( \phi \), the partition function is

\[ Z[\beta] = \text{Tr}(e^{-\beta \hat{H}}) = \int_{\phi(t) = \phi(t+\beta)} \mathcal{D}\phi \ e^{-S[\phi]}. \]  \hspace{1cm} (3.2)

In this compactified time, the space is periodic in one direction, as in \( R^{d-1} \times S^1 \). Next we will construct a model for finite temperature field theory using the gauge/gravity duality.

Let us take the action (2.1)

\[ S = \frac{1}{16\pi G_d} \int d^{d+1}x \sqrt{|g|} \left( R + \frac{d(d-1)}{L^2} \right), \]  \hspace{1cm} (3.3)
with the cosmological constant derived in (2.11). There are two solutions to the Einstein equations (2.2) which have the desired boundary geometry of $\mathbb{R}^{d-1} \times S^1$ and are asymptotically AdS. One is given by the usual AdS solution (2.14) with compactified time

$$d s_E^2 = \frac{L^2}{z^2} d t_E^2 + \frac{L^2}{z^2} d x^2 + \frac{z^2}{L^2} d z^2, \quad \text{with } t_E = t_E + \beta.$$ (3.4)

The other solution is the Schwarzschild-AdS [8] black hole solution

$$d s^2 = \frac{L^2}{z^2}(-f(z) d t^2 + d x^2 + \frac{d z^2}{f(z)}), \quad f(z) = 1 - \left(\frac{z}{z_h}\right)^d.$$ (3.5)

This is also known as the planar black hole. It mathematically resembles the usual Schwarzschild black hole (2.17), but it is asymptotically AdS. It has a horizon at $z = z_h$, where the blackening factor $f(z)$ vanishes. As $z \to 0$, $f \to 1$, the metric becomes the usual AdS metric. The region between $0$ and $z_h$ is outside of the black hole.

Next we will calculate the temperature of the black hole. We can compute the temperature by continuing the metric (3.6) analytically [40]. After a Wick rotation we get

$$d s_E^2 = \frac{L^2}{z^2} \left(f(z) d t_E^2 + d x^2 + \frac{d z^2}{f(z)}\right).$$ (3.6)

Let us focus on the near horizon limit, $z \approx z_h + z'$, of the metric. The Taylor expansion for the blackening factor is

$$f(z) = f(z_h) + f'(z_h) z' + \ldots \approx f'(z_h) z',$$ (3.7)

where we used $f(z_h) = 0$. Now the metric reads

$$d s_E^2 \approx \frac{L^2}{z_h^2} \left(f'(z_h) z' d t_E^2 + d x^2 + \frac{d z'^2}{f'(z_h)}\right).$$ (3.8)

Let us define coordinates

$$\rho = 2 \sqrt{\frac{z'}{|f'(z_h)|}}, \quad \tau = \frac{1}{2} |f'(z_h)| t_E,$$ (3.9)

and we get the metric

$$d s_E^2 \approx \frac{L^2}{z_h^2} \left(\rho^2 d \tau^2 + d \rho^2 + d x^2\right).$$ (3.10)

This metric resembles a cylinder in polar coordinates multiplied by a plane. The cylindrical part has a conical singularity in $\rho$ unless the coordinate $\tau$ is periodic

$$\tau = \tau + 2\pi,$$ (3.11)

which corresponds to an Euclidean periodicity

$$\beta = \frac{4\pi}{|f'(z_h)|}.$$ (3.12)
We need to identify this period in the dual field theory. For this we use equation (??) which states that the bulk metric $g_{ab}$ is related to the boundary metric $g_{\mu\nu}$ by
\[
\lim_{z \to 0} \frac{z^2}{L^2} g_{ab}(x, z) = g_{\mu\nu}(x),
\]
which should be understood such that the $z$ component no longer exists in the boundary metric. From equation (3.6) we see that the boundary metric is simply the Euclidean flat metric
\[
ds_E^2 = dt_E^2 + dx^2,
\]
where the time coordinate is periodic with the period (3.12). We know how to interpret this kind of periodicity in the path integral formalism. The inverse of the period is the temperature of the field theory
\[
T = \frac{1}{\beta} = \frac{|f'(z_h)|}{4\pi} = \frac{d}{4\pi z_h}.
\]

3.1.1 Thermodynamics in gauge/gravity dualities

Now that we have identified the temperature of the field theory, we can study other thermodynamic quantities. We start by investigating the free energy related to the partition function
\[
Z = e^{-\beta F} \implies F = -T \ln Z.
\]
We can calculate the free energy with the duality. The free energy density is natural to consider when the free variables are volume, $V$, temperature, $T$ and particle number, $N$, it is also the one we find from the path integral formalism. We must relate the field theory partition function to the gravity generating functional and its extremals. We find, after some calculation, that
\[
F = TS(g_{ab}),
\]
where the entropy, $S$, for the AdS-Schwarzschild black hole is
\[
S = -\frac{(4\pi)^d L^{d-1}}{16\pi G_d d!} V_{d-1} T^{d-1}.
\]
Here $V_{d-1}$ denotes an infinite volume. For the metric without the black hole the calculation gives $S = 0$. We are also interested in the free energy density of the black hole solution
\[
\mathcal{F} = \frac{F}{V_{d-1}} = -\frac{(4\pi)^d L^{d-1}}{16\pi G_d d!} T^d.
\]
The free energy density of the other solution is zero. Thus the black hole solution at finite temperature always has lower free energy than the thermal AdS. This kind of considerations are important also in the first two papers in this thesis.

We can now calculate the entropy density
\[
s = \frac{S}{V_{d-1}} = -\frac{\partial \mathcal{F}}{\partial T} = \frac{(4\pi)^d L^{d-1}}{16\pi G_d d!} T^{d-1}.
\]
The temperature dependence could also have been obtained from the conformal invariance of the theory. There is yet another way of checking the result using the formula (2.18)

\[ S = \frac{A}{4G_d}, \]  

(3.21)

which yields the same answer as the gauge/gravity models. One famous result of such calculations is the free energy density of \( \mathcal{N} = 4 \) SYM at strong coupling \( \lambda = \infty \) and at weak coupling \( \lambda = 0 \) [35]. We find that

\[ \mathcal{F}(\lambda = \infty) = \frac{3}{4} \mathcal{F}(\lambda = 0). \]  

(3.22)

The result seems to state that thermodynamic properties change only a little even though the coupling varies greatly. Transport coefficients however, change more.

If one also studies the pressure, \( p \), and energy densities, \( \epsilon \), one finds that \( p = \frac{1}{d-1} \epsilon \), so that indeed the stress energy tensor \( T_{\mu \nu} = \text{diag}(-\epsilon, p, \ldots, p) \) is traceless:

\[ T_{\mu \mu} = -\epsilon + (d-1)p = 0. \]  

(3.23)

This is an important consistency check for the dual field theory to be conformally invariant.

The basic methods presented here, suggest that finite temperature conformal and strongly coupled quantum field theories can be mapped into Schwarzschild-AdS black hole backgrounds. In this case it is possible to calculate all the quantities analytically with the use of AdS/CFT. We can also add other symmetries, for example \( U(1) \) symmetry often seen in condensed matter systems. We could do this by including a Maxwell term to the Einstein action (2.1), and once again look for the solutions with some specific ansatz for the Maxwell field. The charged black hole solution turns out to be the Reissner-Nordström-AdS black hole. This model will have two parameters, temperature and the chemical potential, \( \mu \), and physical quantities will depend on their ratio, \( T/\mu \). We will return to these issues in chapter 4.

### 3.2 Improved holographic quantum chromodynamics

This section reviews a recent development in applying gauge/gravity dualities to modeling QCD. Particularly important for the thesis is a model by Kiritsis et. al [9, 41, 42, 43, 44] called improved holographic quantum chromodynamics. The model is defined by a five-dimensional Einstein-dilaton system, with the action

\[ S = \frac{1}{16\pi G_5} \left\{ \int d^5x \sqrt{-g} \left[ R - \frac{4}{3} (\partial \phi)^2 + V(\phi) \right] - 2 \int d^4x \sqrt{-\gamma} K \right\}, \]  

(3.24)

given in the Einstein frame. Here \( G_5 = 1/(16\pi M_p^3 N_c^2) \), \( M_p \) is the five-dimensional Planck constant and \( N_c \) is the number of colors. The first term defines the bulk action and the last part is the Gibbons-Hawking term, \( K \) is the extrinsic curvature of the boundary.

In the model four coordinates \( x_i \) are identified with the 4D coordinates of the boundary and the radial coordinate \( z \) corresponds to the renormalization group scale in the 4D
theory. The dilaton, $\phi$, is identified with the ’t Hooft coupling constant with an unknown proportionality factor, $\kappa$, as

$$\lambda \equiv e^\phi \equiv \kappa N_c g_s^2 g_{YM}.$$  \hspace{1cm} (3.25)

The dynamics of this dilaton is in the dilaton potential $V(\lambda)$. The ultraviolet and the infrared dynamics of the geometry are described by the small-$\lambda$ and large-$\lambda$ asymptotics of the potential. The potential in the original model is

$$V(\lambda) = \frac{12}{L^2} \left( 1 + V_0 \lambda + V_1 \lambda^{4/3} \sqrt{\log (1 + V_2 \lambda^{4/3} + V_3 \lambda^2)} \right).$$  \hspace{1cm} (3.26)

This potential fulfills the requirements for the ultraviolet and infrared dynamics [9], which are:

- For small $\lambda$, that is the ultraviolet limit we demand asymptotic freedom with logarithmic running. The potential is required to have a weak-coupling expansion

$$V(\lambda) \approx \frac{12}{L^2} (1 + v_0 \lambda + v_1 \lambda^2 + \ldots), \quad \lambda \to 0.$$  \hspace{1cm} (3.27)

The $\lambda = 0$ sets the scale $L$, and $V$ is constrained to be finite and positive. The other terms $v_i$ fix the $\beta$-function terms for the coupling $\lambda$. The $v_i$ are related to perturbative $\beta$-function of QCD. After identifying the energy scale with the metric scale factor

$$\beta(\lambda) \equiv \frac{d\lambda}{d \log E} = -b_0 \lambda^2 - b_1 \lambda^3$$

$$b_0 = \frac{9}{8} v_0, \quad b_1 = \frac{9}{4} v_1 - \frac{207}{256} v_0.$$  

- For large $\lambda$, or infrared asymptotics, we require confinement and this forces the potential to be of the form

$$V(\lambda) \approx \lambda^2 Q (\log \lambda)^P, \quad \begin{cases} 2/3 < Q < 2\sqrt{2}/3, & P \text{ arbitrary} \\ Q = 2/3, & P > 0. \end{cases}$$  \hspace{1cm} (3.28)

The case of $Q = 2/3$ and $P \geq 0$ gives asymptotically correct glueball spectrum, $m_n^2 \sim n$. For the model discussed, these values of $Q$ and $P$ are assumed.

In the large-$N_c$ limit, the partition function of this model can be approximated as a sum over classical solutions of the equations of motion

$$Z \approx e^{-S_1(\beta)} + e^{-S_2(\beta)} + \ldots$$  \hspace{1cm} (3.29)

where the indices indicate different saddle points. There is a fixed temperature $T = 1/\beta$, like in section 3.1. In this scenario we have two kinds of euclidean solutions [44]. They are:

1. Thermal gas solution,

$$ds^2 = b_0(z)^2 (dz^2 + dt^2 + dx_m dx^m), \quad \phi = \phi_0(z),$$  \hspace{1cm} (3.30)

with $z \in (0, \infty)$, and
2. Black hole solutions,

\[ ds^2 = b(z)^2 \left( \frac{dr^2}{f(z)} + f(z)dt^2 + dx_m dx^m \right) \phi = \phi(z), \tag{3.31} \]

with \( z \in (0, z_h) \), and \( f(0) = 1 \) and \( f(z_h) = 0 \).

These solutions have clear dual descriptions. The thermal gas background is dual to a confining phase \([43, 44]\). The black hole solutions are dual to a deconfined phase, because the string tension is zero at the horizon and the Polyakov loop has non-zero expectation value \([31]\). The functions for \( b, \phi \) and \( f \) are solved from the equations of motion

\[
\begin{align*}
6 \frac{\dot{b}^2}{b^2} + 3 \frac{\ddot{b}}{b} + 3 \frac{\dot{b} \dot{f}}{f} &= \frac{b^2}{f} V(\phi), \\
6 \frac{\ddot{b}}{b^2} - 3 \frac{\dot{b}}{b} &= 4 \frac{\dot{\phi}^2}{3}, \\
\frac{\ddot{f}}{f} + 3 \frac{\dot{b}}{b} &= 0.
\end{align*}
\]

These equations often require numerical methods. In the papers I and III most of the analysis was done with numerical solutions.

Similarly to what we calculated in section 3.1, the thermodynamics in the deconfined phase is calculated using the black hole solutions and their thermodynamics. The free energy, \( F = E - TS \), entropy, \( S \), and energy, \( E \), are then calculable.

At any temperature, \( T \), the thermal gas solution always exists and different black hole solutions can exist. To find out which solutions dominate the partition function (3.29), we need to calculate the free energy of each solution and find the smallest of them.

The phase structure of the model is \([43, 44]\)

1. There is a minimum temperature, \( T_{min} \). The black hole solutions exist only above this temperature.

2. There can be several branches of black hole solutions, in the specific model of Kiritsis et al., only two exist. Their free energy is larger than the thermal gas free energy at low temperatures and they do not dominate.

3. At \( T = T_c > T_{min} \) there is a first order phase transition to a black hole phase and the system remains in this phase for all \( T > T_c \).

This structure is dependent of the chosen potential, especially the number of black hole branches can differ for different potentials.

The potential (3.26) has many parameters. However, they are not all independent. The ultraviolet expansion constrains, for example, \( V_0 \) and \( V_2 \) \([45]\). Thus only two parameters \( V_1 \) and \( V_3 \) are free. These are then fixed by thermodynamical quantities, such as latent heat and the pressure at a specific value of temperature above the critical temperature. For Kiritsis et. al, this value was \( T = 2T_c \).
Figure 3.1: A schematic plot of the free energy in equation (3.32). The two curved solid lines are the black hole solutions. The vertical dashed line is at $T = T_{min}$ below which the black hole solutions do not exist. At $T = T_c$ the other black hole solution begins to dominate the ensemble. The thermal gas coincides with the horizontal axis, because its free energy density is zero.

The free energy of the solutions has to be integrated numerically for different pairs of $V_1$ and $V_3$. For this we need to solve the entropy and temperature of the solutions and calculate numerically

$$F = \int_{\lambda_h}^{\infty} d\lambda S(\lambda) \frac{dT(\lambda)}{d\lambda}. \quad (3.32)$$

For details, see [46]. The phase structure is also plotted in the figure 3.1 for the potential (3.26). The best fit is obtained by choosing

$$V_1 = 14, \quad V_3 = 170. \quad (3.33)$$

For these values the existence of a critical temperature and the minimum temperature, as defined above, is shown in the plots of free energy and other thermodynamic quantities, figure 3.2. The phase transition at the critical temperature is a first order transition and the latent heat, $L_h$, per unit volume is proportional to the jump in the entropy density at the phase transition:

$$L_h = T \Delta s \approx T_c s(\lambda_c). \quad (3.34)$$

The fits for the coefficients (3.33) are obtained by comparing ratios of $p/T^4$, $e/T^4$ and $s/T^3$, where $p$ is pressure, $e = p + Ts$ is the energy density, to lattice data [47].

Measuring the conformality of the model is also of interest. This is done by calculating the trace anomaly

$$T_\mu^{\mu} = \frac{e - 3p}{T^4}, \quad (3.35)$$

which is the condition (2.26) for scale invariance. This quantity is indeed nonzero, and is plotted for a special case in figure 3.2. The term is related to a boundary operator, the gluon condensate. To obtain the precise relation, one needs to solve the asymptotic behavior of
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Figure 3.2: Energy density, $\epsilon$, pressure, $p$, and the trace anomaly plotted. The plot is taken from the Paper II. It illustrates how the trace anomaly becomes smaller as $T$ increases and also at $T = T_c$.

the functions $b$, $\lambda$ and $f$ in the model and then regulate the free energy, that is equivalently, the Euclidean action evaluated on the solution. In [44] this is calculated in detail using a subscription prescription to avoid calculating the counter term explicitly.

3.2.1 The spatial string tension

The spatial string tension, $\sigma_s$, is a quantity in the spatial Wilson loop, $W(X,Y)$:

$$W(X,Y) = \langle e^{i \oint_{x,y} d x^\mu A^\mu} \rangle \sim e^{-\sigma_s X Y},$$

(3.36)

where the last approximations represents an area law behavior, which is true at high temperatures in SU(2) [48]. The spatial string tension was calculated in the improved holographic QCD by us in Paper I [21]. It has also been calculated on the lattice [48, 49, 50, 51, 52] and with high-temperature perturbative expansion in three-dimensional Yang-Mills theory [53, 54]. The agreement with our result and the lattice is good at temperatures close to the phase transition at $T = T_c$, see figure 3.3

A first-principle method for determining the spatial string tension in QCD matter is to measure rectangular Wilson loops, $W(X,Y)$ of size $X \times Y$ in the $(x,y)$ plane. The potential at a separation, $D$, is defined by

$$V(D) = -\lim_{Y \to \infty} \frac{\log W(D,Y)}{D}$$

(3.37)
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Figure 3.3: The plot of the $T/\sqrt{\sigma_s}$. Thick points are the lattice points, dashed curve is an analytic QCD prediction, the smaller points are the result of the numerical integration of the gravity equations derived in the text, and the continuous curve is the leading approximation. After $T \approx 10T_c$ the QCD prediction grows faster than our result. The figure is from Paper I.

and the spatial string tension by

$$\sigma_s(T) = \lim_{D \to \infty} \frac{V(D)}{D}. \quad (3.38)$$

We take a simplified potential in [21]. The potential we had was

$$V(\phi) = \frac{12}{L^2} (1 + V_1 \lambda^{4/3} \sqrt{\log(1 + V_3 \lambda^2)}), \quad (3.39)$$

with

$$V_1 = 14.3, \quad V_3 = 170.4. \quad (3.40)$$

We also perform an analytic approximation with a yet simpler function. This is done mainly for purposes of gaining analytic understanding of the model. The function is

$$V = \frac{12}{L^2} \left(1 + \frac{8\beta_0}{9(q-1)} \lambda^{q-1}\right). \quad (3.41)$$

To derive the spatial string tension we used the equations for the extremal string configuration. The loop was calculated by setting the end points of the string at $x = \pm L/2$ and the string hanging in the fifth dimension, such that the maximum distance from the boundary is $z_* < z_h$. In short-hand notation

$$b_s(z) = b(z)\lambda^{2/3}(z), \quad b_s = b_s(z_*) \quad (3.42)$$

we found for the spatial loop a relation for $D$ and $z_*$ as

$$D = 2 \int_0^{z_*} \frac{dz}{\sqrt{f(z) [b_s^4(z)/b_s^4 - 1]}} \quad (3.43)$$
and for the potential

\[ V(D) = \frac{1}{\pi \alpha'} \int_e^{z_0} dz b_s^2(z) \frac{1}{\sqrt{f(z)[1 - b_s^2/b_s^*(z)]}}. \] (3.44)

These expressions are plotted in figure 3.4.

The results we obtain together with other calculations in the same model agree with literature fairly well.

### 3.2.2 Thermodynamics in a gauge/gravity dual

The analysis of paper II is based on the improved holographic QCD. Here we put emphasis on the \( \beta \)-function. We take the starting point of a boundary theory with some running coupling \( g^2(\mu) \) with a known \( \beta \)-function

\[ \mu \frac{dg(\mu)}{d\mu} = \beta(g). \] (3.45)

As before we identify the scale \( \mu \) with the function \( b(z) \) in such a way that large energy scales, i.e. the ultraviolet, correspond to \( z \to 0, b(z) \sim 1/z \to \infty \) and small scales to large \( z \). As before, we have the three Einstein equations, and one equation relating the metric function \( b(z) \) and the \( \beta \)-function, \( bd\lambda/db = \beta(\lambda) \).

In the paper we take an extreme point of view and use the two Einstein equations not containing \( V \) and the equation \( bd\lambda/db = \beta(\lambda) \) to completely solve the problem and take the remaining Einstein equation as simply an equation determining \( V \). We concentrate on an ansatz of the form \( \beta(\lambda) = -\beta_0 \lambda^q \). The exponent \( q \) is a parameter, \( q \geq 1 \) corresponds to an IR confining theory, \( q = 2 \), to a one-loop Yang-Mills running and features \( q = 10/3 \) are plotted in figure 3.2.

It is interesting to see that many main features of a first order transition are reproduced by a relatively simple ansatz. A more detailed agreement with lattice data and perturbation theory requires us to use a more complex ansatz \(-3\lambda/2(1 + \alpha/\log \lambda + \ldots)\), which produces a continuous transition for \( \alpha = 0 \) and we recover the first order transition for \( \alpha > 0 \).

This investigation gives us ideas to expand the investigation to different directions and also gives us some analytic insight into the model.
Figure 3.4: Top: Extremal string configurations for different values of $z_\star$. Bottom: The potential $V$ in equation (3.44) plotted as a function of $L$. In the potential plot we have also shown the corresponding values of $z_\star$. 
Chapter 4

Applications to condensed matter

This section of the thesis first discusses the solution of the well-known example of a Maxwell field in the bulk, and then proceeds to review the results of paper III about quantum Hall transitions. We will follow [8, 7, 55].

4.1 Maxwell fields in the duality

We are interested in describing a global $U(1)$ symmetric theory with holographic methods. For this we need to have a finite charge density created by a nonzero chemical potential. In the Lagrangian language this means adding a term $\mu J^t$, where $\mu$ is the chemical potential and $J^t$ is the charge density for the global $U(1)$ symmetry in the field theory. This operator $J^t$ is dual to the time component of the Maxwell field $A_t$ in the bulk. The bulk fields obey Maxwell’s equations and are connected to the field theory quantities as:

$$A_t(z) = \mu + \langle J^t \rangle z + \ldots$$  \hspace{1cm} (4.1)

This shows that the chemical potential and charge density are the boundary values of the Maxwell potential and electric flux,

$$\mu = \lim_{z \to 0} A_t, \quad \langle J^t \rangle = \lim_{z \to 0} F_{zt}. \hspace{1cm} (4.2)$$

The latter equation tells that the bulk has to have a non-vanishing electric flux at infinity in order to have the boundary theory at nonzero charge density.

To investigate this further we set up a simple theory modeled by the action

$$S = \int d^{d+1}x \sqrt{|g|} \left[ \frac{1}{16\pi G_d} \left( R + \frac{d(d-1)}{L^2} \right) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right], \hspace{1cm} (4.3)$$

where $F_{\mu\nu}$ is the electromagnetic field strength, and $g$ the coupling for the field. This action is similar to (2.1) but the last term is new.

We must solve the following equations of motion

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \frac{d(d-1)}{2L^2} g_{\mu\nu} = \frac{(16\pi G_d)^2}{2g^2} (2F_{\mu\sigma}F^{\sigma\nu} - \frac{1}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}), \hspace{1cm} (4.4)$$

$$\nabla_{\mu} F^{\mu\nu} = 0. \hspace{1cm} (4.5)$$
Because we want to have a $U(1)$ field, we will take as an ansatz

$$A_\mu = A_t dt. \quad (4.4)$$

The solutions to these equations are well-known. One finds the AdS-Reissner-Nordström black hole

$$ds^2 = \frac{L^2}{z^2} \left( -f(z) dt^2 + \frac{dr^2}{f(z)} + dx^i dx_i \right), \quad (4.5)$$

where the blackening factor is

$$f(r) = 1 - \left( 1 + \frac{z_h^2 \mu^2}{\gamma^2} \right) \left( \frac{z}{z_h} \right)^d + \frac{z_h^2 \mu^2}{\gamma^2} \left( \frac{z}{z_h} \right)^{2(d-1)}$$

$$\gamma^2 = \frac{(16\pi G_d)^2 (d-1) g_2^2 L^2}{(d-2)}.$$

The $U(1)$ gauge field is

$$A_t = \mu \left[ 1 - \left( \frac{z}{z_h} \right)^{d-2} \right], \quad (4.6)$$

which has the desired feature that $A_t \rightarrow \mu$ as $z \rightarrow 0$, when compared to equation (4.1).

We can calculate the temperature of the solution

$$T = \frac{1}{4\pi z_h} \left( d - \frac{(d-2) z_h^2 \mu^2}{\gamma^2} \right). \quad (4.7)$$

We can compare this to the case of zero chemical potential, equation (3.15), $T = d/(4\pi z_h)$. The case with chemical potential gives freedom to scale out the radial variable but we are not able to scale the chemical potential away. Therefore we are left with the dimensionless ratio of $T/\mu$, which is a tunable parameter. In a scale invariant theory this is now the only parameter.

### 4.2 Transport coefficients

Let us see how we can solve transport coefficients in this setup. We will focus on computing electric currents and conductivities in a 3-dimensional theory with finite chemical potential. To be completely general, we could generalize Ohm’s law to include heat current as well as electric\(^1\), but we will not focus on that in this thesis. Therefore we are interested only in the equation

$$J_i(\omega) = \sigma_{ij}(\omega) E_j(\omega), \quad (4.8)$$

where $J_i$ is the electric current, $\sigma_{ij}$ is the conductivity tensor, $E_i$ is the electric field, and $\omega$ the frequency. It can be shown that the conductivities are proportional to Green’s functions. This means that the off-diagonal conductivities in time symmetric configurations vanish. In the paper III a time antisymmetric situation due to the presence of a magnetic field is

\(^1\)The thermoelectric response is called the Nernst effect
considered. Now in Fourier space we can relate the source $E_j$ to a background value for a vector potential and metric fluctuation. At zero momentum we have

$$ E_j = -\partial_t A_j = i\omega \delta A_j(0). \quad (4.9) $$

In linear response theory the relation between the source and expectation value tells us that to find the conductivity, we need to find the retarded Green's function

$$ \sigma_{xx}(\omega) = -\frac{iG_R^{xx}(\omega)}{\omega}. \quad (4.10) $$

Methods to calculate Green's functions in AdS/CFT were first presented in [56] and then in greater detail [57, 58]. To this end we need to determine the response to the small background field, $\delta A_j$. At zero momentum this is straightforward to do with the background solution (4.5). We linearize the equations of motion about this background and get a single second order differential equation for $\delta A_j$,

$$ (f\delta A'_j + \frac{\omega^2}{f}\delta A_j - \frac{4\mu^2 z^2}{\gamma^2 z_h^2} \delta A_j = 0. \quad (4.11) $$

The solutions to this equation can be expanded in a series near $z = 0$ as

$$ \delta A_j = \delta A_j(0) + \frac{z}{L} \delta A_j(1) + \ldots. \quad (4.12) $$

Here the $\delta A_j(i)$ are the constants of integration of the second order linear differential equation we are solving. To be able to solve (4.11), we first expand the equation near the horizon, and then use a near horizon expansion to determine initial values for the function and its differential near the horizon. Then we use these initial values and solve the complete equation. The proportionality of the $A_j$ terms near the boundary is $\omega$-dependent and we extract it from the numerical solution. Using knowledge from the correspondence we can find the conductivity, with the equation (2.54),

$$ \sigma(\omega) = -\frac{1}{g^2 L} \frac{i}{\omega} \frac{\delta A_j(1)}{\delta A_j(0)}. \quad (4.13) $$

For the numerical solution we need to take care of the singularity of the equation (4.11) at the horizon. This is done by expanding the background around the horizon, $z_h$, to set the initial values at some small distance from the horizon, $\epsilon = z - z_h$. It is also useful to scale the perturbation

$$ A_j(z) = f(z)^{-\frac{i}{4\pi \omega / T}}(1 + A_1 \epsilon + A_2 \epsilon^2 + \ldots). \quad (4.14) $$

The equation becomes regular and it can be numerically solved to find the boundary solution. After this, we can separate the coefficients and plot the conductivity, see figure 4.1. More discussion on the boundary conditions can be found in [8, 59]

These plots have been compared with actual data. It has been suggested that the experimental data in graphene might show similar behavior as these plots. Graphene is a material, which is described by a $2+1$ dimensional relativistic theory with a chemical potential, for a
Figure 4.1: The real and imaginary part of the conductivity. Blue dots are the real part and it asymptotes to one and the grey dots are the imaginary part, which asymptote to zero. The plots shown here were produced with Mathematica with the method described in the text.

review, see [60]. This makes it an appealing candidate and it has been widely investigated [61, 62, 63, 64].

We can expand our results such that there would be a magnetic field in the background. The Green’s functions are not so restricted anymore and the conductivity can have off-diagonal, i.e., Hall conductivity, terms. The most general Ohm’s law now reads including thermoelectric response

\[
\left( \begin{array}{c} \langle J_i \rangle \\ \langle Q_j \rangle \end{array} \right) = \left( \begin{array}{cc} \sigma_{ij} & \alpha_{ij} T \\ \alpha_{ij} T & \kappa_{ij} T \end{array} \right) \left( \begin{array}{c} E_j \\ -\langle \nabla_j T \rangle / T \end{array} \right). \tag{4.15}
\]

Using similar calculations, it has been shown that in this case [55, 65], the conductivity is, in a certain regime of validity, given analytically by

\[
\sigma_{xx} = \sigma_Q \frac{\omega (\omega + i \gamma + i \omega_c^2 / \gamma)}{(\omega + i \gamma)^2 - \omega_c^2},
\]

\[
\sigma_{xy} = -\frac{\rho B}{\omega_c} \frac{-2 i \gamma \omega + \gamma^2 + \omega_c^2}{(\omega + i \gamma)^2 - \omega_c^2},
\]

where \( \omega_c = B \rho / (\epsilon + P) \) and \( \gamma = \sigma_Q B^2 / (\epsilon + P) \), \( \sigma_Q \) is a constant [66].

We can investigate some limits of these solutions. First we consider the DC limit, \( \omega \to 0 \), which gives

\[
\sigma_{xx} = 0,
\]

\[
\sigma_{xy} = \frac{\rho}{B},
\]

this result gives us the Hall conductivity at zero frequency. It is proportional to the charge density. This tells us that we need positive or negative charges to get a Hall conductivity. Physically, one can think that under constant magnetic and electric fields, positive and negative charges move in the same direction and their net current is thus zero.
Figure 4.2: Longitudinal conductivity $\sigma_{xx}$ and Hall conductivity $\sigma_{xy}$ a) from the model discussed in the text and b) from Sachdev’s field theoretic model. Real parts are depicted by the solid curves and imaginary parts by the dashed lines. For the plots in a the parameters of the model had values $\rho = 15.1$, $B = 1.4$ and $\tau = 0.3$.

### 4.3 Quantum Hall transitions

In paper III [67] we consider AdS dual descriptions of quantum phase transitions corresponding to transition between different quantum Hall plateaus, modeled by a system of interacting branes. The branes are set up in such a way that they have a single mutual transverse direction, and the transitions are realized by taking the separation of the branes to zero and then continuing to the other side.

We calculate analytically DC electrical conductivities valid for a wide class of transitions. In the middle of the Hall plateau-plateau transition we compute the longitudinal and transverse AC conductivities at finite temperature. These results are plotted in figure 4.2, where Sachdev’s results from a simplified field theoretic model are also plotted for comparison. We find a good agreement for certain values of the parameters in our model. Our results are unexpectedly good at least when comparing to the complexity of the calculations needed in the model considered by Sachdev.

For the brane system we arrive at Born-Infeld action of the form

$$S = -\int d^4x \tau(r) \sqrt{-\det(g_4 + F)}.$$  \hspace{1cm} (4.16)

We then expand this action to quadratic fluctuations about a background solution. The Born-Infeld part of the action then reads

$$S = \frac{1}{2} \int d^4x \frac{\tau(r)}{\sqrt{\Delta_0}} \left[ 2Vf_{ri}f_{vi} - 2BF^{(0)}_{ri} \epsilon_{ij}f_{vi}f_{vj} + UVf_{ri}f_{ri} \right],$$  \hspace{1cm} (4.17)
where the functions $U(r)$ and $V(r)$ are at this stage some general functions compatible with describing asymptotically AdS black brane. In this expression,

$$\Delta_0 = (B^2 + V^2)(1 - (F_{rv}^{(0)})^2), \quad (4.18)$$

and $F_{rv}^{(0)}$ is a background solution and the small $f_{ij}$ correspond to fluctuation of the gauge fields

$$A_\mu = A_\mu^{(0)} + a_\mu. \quad (4.19)$$

There is also a Chern-Simons term included in the complete action.

We consider a complex combination of the $(a_v, a_i)$ with a harmonic time dependence $e^{-i\omega v}$

$$a_\pm = a_x \pm ia_y \quad (4.20)$$

leading to

$$C\partial_r^2 a_\pm + (C' - 2i\omega)\partial_r a_\pm - i\omega(A' \mp iB')a_\pm = \pm \frac{k'(r)}{2\pi \tau_\infty} \epsilon_{ij} f_{ij}. \quad (4.21)$$

All the functions $A, B, C,$ and $k$ are explained in detail in [67]. From here we solve for the asymptotic behavior

$$a_\pm = a_\pm^{(0)} + \frac{1}{r} a_\pm^{(1)} + \ldots \quad (4.22)$$

which is then used to solve for the conductivities

$$\sigma_\pm(\omega) = \left(1 - i \frac{a_\pm^{(1)}}{\omega a_\pm^{(0)}}\right) \tau_\infty. \quad (4.23)$$

Using this knowledge we were able to extract numerically the AC conductivity and obtain analytical expression for the DC limit.
Chapter 5

Conclusions

Strongly interacting quantum field theories are difficult to solve. The AdS/CFT correspondence gives a method for analyzing strongly interacting theories using classical supergravity. It is a realization of the holographic principle as it relates a higher dimensional gravitational theory to a field theory living on the boundary of the bulk.

In chapter 2, we reviewed the ingredients of the correspondence, anti de Sitter spaces and conformal field theories, before venturing on to the correspondence. We also touched upon the thermodynamics of black holes. We derived a relation for the field theory temperature and the temperature of a black hole solution in the bulk, and showed how to compute expectation values of operators, in particular, the example of scalar field was solved.

Chapter 3 dealt with a specific application to particle physics. We introduced the improved holographic QCD model and reviewed its basic consequences, such as the general phase structure of the model, entropy, and free energy. Two subchapters introduces the results of papers I and II in this thesis.

Chapter 4 presented a well-known example of Maxwell fields in the bulk. The motivation to add the $U(1)$ field is to get a finite charge density. First we reviewed how to calculate the temperature and how the Maxwell field is related to the chemical potential. The results then were used to express the conductivity of the model. We described also in detail how to solve numerically the linearized equation for the Maxwell field. These methods were applied in paper III of this thesis.
Bibliography


