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Trace Anomaly
in Semiclassical Quantum Gravitation
and its Applications
in the Problem of Dark Energy

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In this thesis we give a self-sufficient introduction to the trace anomaly and its applications in the problem of cosmological constant. We begin by revising the renormalization of quantum electrodynamics in flat space and the Lagrangian formalism of general relativity. Then we discuss shortly about the renormalizability of quantum general relativity, after which we turn our attention to a semiclassical theory of quantum gravitation.

We review the construction and renormalization of the semiclassical theory, and discuss shortly the stability of it. We then proceed to examine the trace anomaly of the semiclassical theory, and begin by reviewing Weyl cohomology in n-dimensions. We use the Weyl cohomology to construct the Wess-Zumino action, from which we derive a non-local action for the trace anomaly. The non-local action is then rendered local by introducing new auxiliary fields, in which the non-local behaviour of the action is contained.

After all these theoretical considerations we finally examine the non-trivial cosmological consequences of the trace anomaly. At first we review shortly the Friedman-Robertson-Walker-model and its classical perturbations, after which we examine the linear perturbations of the trace anomaly action in de Sitter space. We find that when the auxiliary fields of the action are quantized, the cosmological constant becomes dependent on the border conditions at the horizon scale of de Sitter space. We then conclude that the small but non-zero value of the cosmological constant could be a physical consequence of the presence of the horizon.
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Notation

Riemannian geometry

The sign conventions for the metric and curvature tensors are (- - -) in the terminology of Misner, Thorne and Wheeler. That is, we use the metric signature (+ - -), write the Riemann tensor as

\[ R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\alpha\rho} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma} \Gamma^{\alpha}_{\nu\rho} \], \hspace{1cm} (0.1)

and define the Ricci tensor to be \( R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \). In curved space-time we denote the metric with \( g_{\mu\nu} \) and as a special case the Minkowski metric is denoted with \( \eta_{\mu\nu} \). The determinant of the metric is denoted with \( g \equiv \det(g_{\mu\nu}) \). \hspace{1cm} (0.2)

We shall also use Einstein summation convention and use \( \alpha, \beta, \gamma \ldots \) for the indices to be summed and \( \mu, \nu, \rho, \sigma \) for indices not to be summed. We will also write the partial derivatives with a comma before the indices and the covariant derivatives with a semicolon

\[ R_{\mu} \equiv \partial_{\mu} R = \frac{\partial}{\partial x^{\mu}} R \hspace{1cm} (0.3) \]
\[ R_{\mu} \equiv \nabla_{\mu} R \hspace{1cm} (0.4) \]

The symmetrization between two indices is denoted by parenthesis

\[ T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \hspace{1cm} (0.5) \]

and the antisymmetrization by square brackets

\[ T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \hspace{1cm} (0.6) \]

The contraction of a vector with gamma matrices is denoted with the Feynman slash

\[ \phi = \gamma_{\alpha} a^{\alpha} \hspace{1cm} (0.7) \]
Lagrangian formalism

Birrell & Davies [5] use two different Lagrangians $\mathcal{L}$ and $L$, and the action

$$ S = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}L . $$

We will take the square root to be a part of the integration measure and define the Lagrangian and action as

$$ S = \int d^4x\sqrt{-g}\mathcal{L} . $$

Units and abbreviations

Most of the time we use the natural units with $c = \hbar = \epsilon_0 = 1$, the only notable exception being section 2.3 where we expand the Einstein field equations around $\hbar$. The following abbreviations are also used:

- $^*$ complex conjugate
- $^\dagger$ Hermitian conjugate
- $\bar{}$ Dirac adjoint
- $\partial_\mu$ partial derivative
- $\nabla_\mu$ covariant derivative
- $\Box = g^{\alpha\beta}\nabla_\alpha \nabla_\beta$ generalized d’Alembert operator
- $\ln$ natural logarithm
- $\text{tr}$ trace
- $\text{diag}$ diagonal matrix
- $\sim$ order of magnitude estimate
- $\equiv$ equal by definition
Introduction - The Problem with the Cosmological Constant

The origins of the cosmological constant $\Lambda$ date back to the beginning of the 20th century, when Einstein proposed his model for the universe. He based his theory on the idea that the universe is unchanging and added to the theory a cosmological constant. In the resulting solution, called the Einstein universe, the cosmological constant prevents the universe from expanding or collapsing. The cosmological constant effectively creates a negative, constant pressure, which exactly cancels the attraction of the matter in the universe rendering it stable.

In 1927 Georges Lemaître derived the Hubble law and proposed that the universe might be expanding [9]. Two years later Edwin Hubble confirmed the law that now bears his name from the redshift of extra-galactic nebulae [8]. The notion of expanding universe lead quickly to the demise of the static Einstein universe, which was superseded by the Friedman-Robertson-Walker universe. The Friedman-Robertson-Walker universe describes expanding or contracting universe without any cosmological constant making it thus redundant.

The situation changed again in 1998, when observations of type Ia supernovae suggested that the expansion of the is accelerating [10]. The acceleration of the expansion was an unexpected observation as the matter-energy content of the universe can only slow down the expansion rate of the universe. This lead to a new rise of the cosmological constant, which was now used to explain the acceleration. With sufficiently large cosmological constant the negative pressure would surpass the gravitational attraction leading to an accelerating expansion. The cosmological constant needed to explain the current observations is of the order

$$\Lambda \sim 10^{-52}m^{-2},$$

(0.10)
a small, yet non-zero positive constant.

Soon after the invention of the cosmological constant it was hypothesized that it would be related to vacuum energy of the quantum field theories. Just as the cosmological constant, the vacuum energy of an undisturbed vacuum is evenly distributed in the space. Hence it is a good candidate for the source of the cosmological constant. However, there is a catch: in classical physics there is no natural scale for $\Gamma$. If the reduced Planck constant is zero $\hbar = 0$, there is no fixed length scale in the Einstein field equations at all, and any value of $\Lambda$ is possible. Things change when we let $\hbar \neq 0$, as there appears a quantity
with the dimension of length that can be formed from the constants present in general relativity, the Planck length

\[ L_{\text{pl}} \equiv \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{m}. \]  

(0.11)

When the quantum theories are considered in a general relativistic setting, there is a dimensionless pure number

\[ \lambda \equiv \Lambda L_{\text{pl}}^2 = \frac{\hbar G \Lambda}{c^3} \sim 10^{-122}, \]  

(0.12)

called the dimensionless cosmological constant, which value one might expect to be addressed by the quantum theory. The value of \( \lambda \) is so small that its determination has acquired the questionable title as the worst finite-tuning problem in the history of physics.

The vacuum energy of a quantum field theory depends greatly on the short wavelength ultraviolet fluctuations of the quantum fields. These fluctuations will in general lead to a large contribution to the cosmological constant. In fact the dimensionless cosmological constant obtained from the ultraviolet estimates of the field theories is proportional to unity, which means that the order of magnitude is \( 10^{122} \). Undoubtedly that makes the ultraviolet estimate the worst approximation in the history of physics.

The failure of the ultraviolet estimates has lead the debate to semiclassical quantum gravity. Although we don’t have a complete quantum theory for the gravity, a low energy effective theory might at least give some hint of the origin of the cosmological constant. The idea of using the semiclassical theory to produce the cosmological constant was first brought up by Starobinsky in the context of cosmic inflation \([17]\). In Starobinsky’s model the trace anomaly of the semiclassical theory produces a cosmological term, which drives the inflation of the young universe. The trace anomaly is a spontaneously generated quantum anomaly, that is related only to the number of different quantum fields in the space. Thus it could generate the cosmological term without any fine-tuning. However, the cosmological constant obtained from the trace anomaly without any further modifications is too large, the order of magnitude being \( \lambda \sim 10^{-10} \).

The trace anomaly will add new auxiliary fields to the semiclassical theory, which as new degrees of freedom. These auxiliary fields are sensitive to the global border conditions on the cosmological horizon scale. It has been recently proposed by Emil Mottola and others in \([1]\), \([3]\), \([10]\), that the auxiliary fields should be quantized. When these fields are allowed to fluctuate freely, the cosmological constant seems to become scale dependent so that it decreases as the distances increase. This would certainly explain the small value of \( \lambda \) without any fine-tuning.

In the end we should note that the cosmological constant and vacuum energy are not the only theory for explaining the accelerating expansion. Most notable of these alternative explanations are \( f(R) \) gravity, phantom energy, quintessence and dark fluid. The \( f(R) \) gravitation extends the general relativity by adding new terms into the Einstein-Hilbert action, which generates the
field equations. This approach doesn’t require any exotic form of energy, but many of the possible extension terms has already been ruled out.

The phantom energy and quintessence are close relatives to the standard dark energy, from which they differ by the modified equations of state. For the standard dark energy the equation of state is \( p = -\rho \), while for the phantom energy \( p < -\rho \) and for the quintessence \( p = w(Q, V(Q))\rho \), where \( p \) is the pressure and \( \rho \) is the density of the energy, and \( w(Q) \) is function of the kinetic and potential energy of the universe. Basically the phantom energy leads to unbounded acceleration, while the acceleration caused by quintessence changes over time.

The dark fluid goes one step further and assumes that the space itself acts like a fluid. The dark fluid approach produces many other gravitational theories as special cases including quintessence and \( f(R) \), and explains also other gravitational phenomenons than the cosmological constant. None of the mentioned alternative approaches can be ruled out by the current observations, and they all have their strengths and weaknesses, just as the standard dark energy model.
Chapter 1

General Relativity
and Quantum Field Theory

We begin this chapter by reviewing shortly the Lagrangian formalism and the use of actions in quantum electrodynamics (QED) in Minkowski space. After that we revise the renormalization by dimensional regularization and counterterms in the Lagrangian. This allows us to consider the renormalizability of general relativity (GR) at abstract level and works as a reference point when we begin to build semiclassical theory in the next chapter. We will also discuss shortly about the axial anomaly, which is a suitably simple example of an anomaly in a quantum field theory.

After reviewing the renormalization procedure in QED we shall focus on classical general relativity. We review shortly the Lagrangian formalism in the classical general relativity and the most important properties of the Einstein field equations (EFE). In the end of this chapter we shall finally discuss about quantum general relativity and its renormalization arguing, that complete theory of quantum gravitation is not renormalizable.

1.1 Renormalization of Quantum Electrodynamics

1.1.1 Lagrangians and actions in Quantum Electrodynamics

The core component of the quantum electrodynamics is the gauge covariant Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\slashed{D} - m)\psi - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} ,$$

where the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\psi, A_\mu$ are the electron and photon field. The gauge covariant derivative is defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu ,$$
and allows the interaction between the electron and photon fields. By expanding the derivative and rearranging the terms we see that

$$L_{\text{QED}} = \bar{\psi} \left( i \frac{\partial}{\partial t} - m \right) \psi - e \bar{\psi} A \psi - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} .$$  \hspace{1cm} (1.3)$$

The first term in (1.3) is the kinetic energy of the electron field, while the second represents the interaction between the field and the third term is the kinetic term for the photon field. These terms can then be associated respectively with the electron propagator, the interaction vertex and the photon propagator of the Feynman diagrams.

The action for QED is defined by the Lagrangian

$$S_{\text{QED}} = \int d^4 x L_{\text{QED}} .$$  \hspace{1cm} (1.4)$$

The equations of motion for the particle fields are then obtained from the action by functionally differentiating $S_{\text{QED}}$ with respect to the field. The equation of motion for the electron field is then

$$\frac{\delta S_{\text{QED}}}{\delta \bar{\psi}} = \left[ i \frac{\partial}{\partial t} - m - e A \right] \psi = 0 ,$$  \hspace{1cm} (1.5)$$

and for the photon field we have

$$\frac{\delta S_{\text{QED}}}{\delta A_\mu} = F^{\alpha\mu} \gamma_\alpha - e \bar{\psi} \gamma_\mu \psi = 0 .$$  \hspace{1cm} (1.6)$$

By rearranging the field sources to the right-hand side of the equations we get

$$\left[ i \frac{\partial}{\partial t} - m \right] \psi = e A \psi$$

$$\Box A^\mu = e j^\mu ,$$  \hspace{1cm} (1.7, 1.8)$$

where the vector current is defined as $j^\mu \equiv \bar{\psi} \gamma^\mu \psi$ and we have chosen the gauge so that the photon field is divergence free, $A^\alpha_{,\alpha} = 0$. From (1.8) it is evident that the vector current $j^\mu$ acts as a source for the photon field $A^\mu$.

By taking the divergence of (1.8) we get the vector current conservation equation

$$j^{\alpha}_{,\alpha} = 0 .$$  \hspace{1cm} (1.9)$$

This implies that $j^\mu$ is a Noether current with an associated symmetry of the action

$$\psi \rightarrow e^{i \omega(x)} \psi .$$  \hspace{1cm} (1.10)$$

The conserved quantity of this symmetry is the electrical charge.
1.1.2 The Three Primitive Divergences

Now we shall shortly review the renormalization of QED in Minkowski spacetime. We will loosely follow the treatment given by R. Ticciati in chapter 19 of [19]. The procedure used in this section will transfer into the curved space-time with slight modifications.

The Lagrangian of QED in general gauge is

\[
L_{\text{QED}} = \bar{\psi} \left( i \frac{\partial}{\partial t} - m \right) \psi - \bar{\psi} e \mathcal{A} \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2\alpha} (\partial_{\mu} A_{\mu})^2 ,
\]

where we have added the last gauge fixing term into the Lagrangian of the previous section. The gauge can now be adjusted freely by setting the gauge fixing parameter \( \alpha \), some special cases being the Feynman gauge (\( \alpha = 1 \)) and Landau gauge (\( \alpha = 0 \)). As we mentioned in the previous section, the three first terms of the QED Lagrangian can be associated with the three basic elements of the Feynman diagrams. It is important to note that any Feynman diagram can be constructed from these three basic components.

Let's then consider diagrams with loops. One complex example is given in fig. 1.1.2. If we were to calculate the amplitude corresponding to this diagram we would notice that there is a divergence associated with every single loop in this diagram. These divergences can be tracked down to diverging loop corrections of the three primitive elements given in fig. 1.1.1.

At one loop level (the smallest correction to the tree-level diagrams) we can draw one diverging loop diagram for each of the primitive elements (fig. 1.1.3). If we can now render these three diagrams finite, all the divergences in every Feynman diagram (at one loop level) will vanish. As we noted above, these diagrams can be associated with terms in the Lagrangian and hence we should remove the divergences from the Lagrangian itself before we try to calculate any amplitudes.

1.1.3 Renormalization by counterterms

The actual procedure we shall use to remove the divergences is called renormalization by counterterms. The idea is to make a suitable change of variables in the Lagrangian that allows us to absorb the divergences in some physical constants, which we then set finite by measuring the actual values of these

---

(a) Electron propagator    (b) Interaction vertex    (c) Photon propagator

Figure 1.1.1: The three primitive elements of Feynman diagrams
constants. In practice we will want to split the Lagrangian into finite and divergent part as

\[
\mathcal{L}_0 = \mathcal{L}_{\text{fin}} + \mathcal{L}_{\text{div}} ,
\]

(1.12)

where the lower index zero signifies that we are working with the unrenormalized Lagrangian. The division can be done with the following change of variables

\[
\psi = \frac{\psi_0}{\sqrt{Z_2}} , \quad A^\mu = \frac{A_0^\mu}{\sqrt{Z_3}} ,
\]

\[
m = m_0 Z_2 , \quad \alpha = \frac{\alpha_0}{Z_3} , \quad e = e_0 \frac{Z_2}{Z_1} \sqrt{Z_3} .
\]

(1.13)

Here the zero signifies again that the quantity is unrenormalized and potentially infinite. This transformation is nothing more than a rescaling of the electron’s mass and charge, the fields and the gauge fixing constant with the renormalization constants \(Z_1\), \(Z_2\) and \(Z_3\). After this scaling the divergent part of the Lagrangian takes the form

\[
\mathcal{L}_{\text{div}} = (Z_2 - 1) \bar{\psi} (i \partial) \psi - (m_0 Z_2 - m) \bar{\psi} \psi
\]

\[
- (Z_1 - 1) \bar{\psi} e A \psi - \frac{1}{4} (Z_3 - 1) F_{\mu \nu} F^{\mu \nu} .
\]

(1.14)
We can now associate the renormalization constants with the divergences in the three primitive elements of Feynman diagrams. The first line in (1.14) is associated with the electron propagator  [1.1.1a] and hence the renormalization constant $Z_2$ is determined by calculating the divergent part of the loop-diagram for the electron propagator in fig. [1.1.3a]. The constants $Z_1$ and $Z_3$ are determined similarly from the loop-diagrams of the interaction vertex and the photon propagator.

The renormalized Lagrangian $\mathcal{L}_{\text{ren}}$ is then defined to be the original Lagrangian $\mathcal{L}_0$ minus the divergent part, which is exactly the Lagrangian of QED with rescaled quantities

$$\mathcal{L}_{\text{ren}} \equiv \mathcal{L}_0 - \mathcal{L}_{\text{div}} = \bar{\psi} \left( i\partial_\mu - m \right) \psi - \bar{\psi} e \mathcal{A}_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 .$$

(1.15)

Thus by rescaling the mass and charge of the electron, the fields and the gauge parameter we have removed all the divergences in QED at one loop level.

If we had considered more than one loop, there would have been more divergences coming from the higher order corrections to the three primitive diagrams. However, all these divergences can be removed by rescaling with the three renormalization constants $Z_1$, $Z_2$ and $Z_3$. At any order in the perturbation theory we always need only these three constants to remove every single divergence in every Feynman diagram.

1.1.4 The Axial Anomaly

Now we will discuss one non-trivial consequence of the renormalization procedure, the axial anomaly. Our approach is follows loosely [7], chapter 4.2. We shall begin by considering the unrenormalized QED Lagrangian (1.11). This Lagrangian is invariant under local $U(1)$ transformation

$$\psi \rightarrow e^{i\omega(x)} \psi ,$$

(1.16)

which means that the vector current $j^\mu = \bar{\psi} \gamma^\mu \psi$ is conserved. If the electron was massless the Lagrangian would be invariant under a second transformation

$$\psi \rightarrow e^{i\omega(x) \gamma_5} \psi ,$$

(1.17)

where the fifth gamma matrix is

$$\gamma_5 \equiv i\gamma_0 \gamma_1 \gamma_2 \gamma_3 .$$

(1.18)

This is the axial transformation, which corresponds to the axial current $j^5_\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi$. The classical conservation laws for these currents are

$$j^\mu_{\mu} = 0$$

(1.19)

$$j^5_{\mu} = 2im\bar{\psi} \gamma_5 \psi ,$$

(1.20)

from which we see that both currents are classically conserved provided that $m = 0$. 

5
In QED the classical conservation laws \[(1.20), (1.19)\] will be modified by the quantum effects that arise from the possibility of adding loops to the amplitude diagrams. In order to calculate the corrections we need to consider the triangle diagrams of fig. 1.1.4. For simplicity we consider only the case \(m = 0\). The analytical expression for fig. 1.1.4a is then

\[
T_{\mu\nu\lambda}(k, p) = -\left(-ie\right)^2 \int \frac{d\ell}{(2\pi)^4} \text{tr} \left[ \frac{i(\ell + p)}{(\ell + p)^2} \gamma_\lambda \gamma_5 \frac{i(\ell - k)}{(\ell - k)^2} \gamma_\nu \gamma_\mu \right],
\]

(1.21)
as given in chapter 19.2 of [14]. Taking the divergence of the axial current is now equal to contracting \(T_{\mu\nu\lambda}\) with \(iq^\lambda\), which is not trivial, because dimensional regularization requires us to consider \(D\)-dimensional loop-momentum \(\ell\).

In general dimension \(D\) the fifth gamma matrix anticommutes with \(\gamma_\mu\) for \(\mu = 0, 1, 2, 3\) and commute for \(\mu = 4, 5, 6\)... Hence we divide the loop-momentum into two parts

\[
\ell = \ell_\parallel + \ell_\perp,
\]

(1.22)
so that \(\ell_\perp^\mu = 0\) for \(\mu = 0, 1, 2, 3\) and \(\ell_\parallel^\mu = 0\) for \(\mu = 4, 5, 6\)... Then we have the identity

\[
q^\lambda \gamma_\lambda \gamma_5 = (\ell + p - \ell + k) \gamma_5 = (\ell + k) \gamma_5 + \gamma_5(\ell - p) - 2\gamma_5 \ell_\perp,
\]

(1.23)
and \(iq^\lambda T_{\mu\nu\lambda}\) is

\[
iq^\lambda T_{\mu\nu\lambda}(k, p) = e^2 \int \frac{d\ell}{(2\pi)^4} \text{tr} \left[ \frac{\ell}{\ell^2} \gamma_\nu \gamma_5 \frac{(\ell + k)}{(\ell + k)^2} \gamma_\mu - \gamma_5 \frac{\ell}{\ell^2} \gamma_\mu \frac{(\ell + p)}{(\ell + p)^2} \gamma_\nu - 2 \frac{\ell}{\ell^2} \frac{(\ell + p)}{(\ell + p)^2} \gamma_\perp \gamma_\nu \right],
\]

(1.24)
where we have shifted the integral in the first term $\ell \to \ell + k$, and anticommuted $\gamma_5 \gamma_\nu$ in the second term.

The crossed diagram in fig. [1.1.4b] contributes as $i q^\lambda T_{\nu \mu \lambda}(p, k)$. By examining (1.24) we see that the first two terms in the trace are antisymmetric for the exchange $\mu \leftrightarrow \nu$, $k \leftrightarrow p$ while the third term is symmetric. Thus the first two terms vanish and we have

$$i q^\lambda [T_{\mu \nu \lambda}(k, p) + T_{\nu \mu \lambda}(p, k)] = e^2 \int \frac{d\ell}{(2\pi)^4} \text{tr} \left[ -2 \frac{(\ell + p)}{(\ell + p)^2} \ell \gamma_5 \gamma_\nu \ell \gamma_\mu \right].$$

After some tedious algebra we find that

$$i q^\lambda (T_{\mu \nu \lambda}(k, p) + T_{\nu \mu \lambda}(p, k)) = -\frac{e^2}{2\pi^2} \epsilon_{\mu\alpha\nu\beta} k^\alpha p^\beta.$$  (1.25)

The complete calculation is given in chapter 19.2 of [14].

Because $j_5^\mu$ couples to the photon field, this relation can be translated into an identity for the axial current operator

$$\partial_\mu \langle j_5^\mu \rangle = -\frac{e^2}{(4\pi)^2} \epsilon_{\mu\alpha\nu\beta} F_{\mu\alpha} F_{\nu\beta}.$$  (1.26)

Thus the quantum corrections lead to a non-zero result, the axial anomaly, on the right-hand side of (1.27). Calculation for the axial and vector current in the case $m \neq 0$ is given in [14], chapter 4.2 and the result is

$$\langle j_5^\mu \rangle = 0$$  (1.28)

$$\langle j_5^\mu \rangle = 2im\langle \bar{\psi} \gamma_5 \psi \rangle - \frac{e^2}{(4\pi)^2} \epsilon_{\mu\alpha\nu\beta} F_{\mu\alpha} F_{\nu\beta}.$$  (1.29)

Thus the quantum mechanical corrections lead to an anomalous contribution, the second term in (1.29), in the axial current conservation law.

The axial anomaly has some interesting properties. First we should note that the classical symmetry $\psi \to e^{i\omega(x)\gamma^5} \psi$ is broken in QED because of the axial anomaly. The breaking of classical symmetries is not limited only to QED but is a general property of quantum field theories. Secondly we note that the anomaly is independent of the renormalization procedure, although in the case of dimensional regularization it is produced in a physically appealing way. In the dimensional regularization the anomaly is generated because we are forced to make the calculations in $D$-dimensions where the classical symmetry is broken spontaneously. At the limit $D \to 4$ this symmetry break leads to a finite residue, the axial anomaly, even though the Lagrangian is invariant in four dimensions.

1.2 Classical General Relativity

1.2.1 Lagrangian formalism in General Relativity

General relativity can be built on Lagrangian formalism just like QED. We shall begin our review of general relativity by considering the Einstein-Hilbert
action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right] . \]  

(1.30)

from which the Einstein field equations can be derived (there will be an additional surface term which is always required to vanish). The total Lagrangian can be read straight from the square brackets in (1.30) and is

\[ \mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M , \]  

(1.31)

where \( \mathcal{L}_M \) is the Lagrangian of the matter fields. The first term depends only on the metric and its derivatives while the second term depends on the matter and energy content of the spacetime. Hence the first term is also called the gravitational Lagrangian and denoted with

\[ \mathcal{L}_g \equiv \frac{1}{2\kappa} (R - 2\Lambda) . \]  

(1.32)

The Einstein field equations can be derived from the Einstein-Hilbert action by varying it with respect to the metric (and discarding surface terms, which are required to vanish)

\[ \frac{2\kappa}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) - \kappa T_{\mu\nu} = 0 . \]  

(1.33)

Thus in GR the metric plays the same role as the quantum fields \( \psi \) and \( A^\mu \) in the QED. The stress-energy tensor \( T_{\mu\nu} \) is defined by the Lagrangian of the matter fields

\[ T_{\mu\nu} = -2 \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M . \]  

(1.34)

As a simple example we take the free, non-interacting scalar field for which the Lagrangian is

\[ \mathcal{L}_S = \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 . \]  

(1.35)

The corresponding stress-tensor is then

\[ T_{\mu\nu} = -\phi_{,\mu} \phi_{,\nu} + \frac{1}{2} g_{\mu\nu} (\phi_{,\alpha} \phi^{,\alpha} - m^2 \phi^2) . \]  

(1.36)

Another example would be the electromagnetic field for which

\[ \mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \]  

(1.37)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor. The stress-tensor then takes the form

\[ T_{\mu\nu} = F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} . \]  

(1.38)
1.2.2 Einstein Field Equations

Let’s then consider the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \]

(1.39)

in more detail. The Ricci tensor \( R_{\mu\nu} \) and Ricci scalar \( R \) depend only on the metric and its first and second derivatives, while the cosmological constant \( \Lambda \) is a parameter. Hence EFE is a set of non-linear second order partial differential equations for the metric. Because the metric is symmetric and the Riemann tensor must obey the Bianchi identity, there are six independent components. This leaves the metric with four gauge degrees of freedom, which correspond to the freedom to choose a coordinate system.

We will now define a quantity to be geometrical if it depends only on the metric and its derivatives. The Ricci tensor and Ricci scalar are both such geometrical quantities, and hence the left-hand side of EFE is purely geometrical.

The right-hand side of the EFE consists of the stress-energy tensor multiplied by a constant which we shall denote with

\[ \kappa = 8\pi G \] .

(1.40)

As we saw in the previous section, the stress-energy tensor is determined by classical fields which represent the matter and energy content of the spacetime. It depends also on the metric but not on any of its derivatives and can thus be regarded as a source for the left-hand side, just as the vector current \( j^\mu \) in QED acts as a source for the photon field.

By taking the divergence of the EFE we see that

\[ R^{\mu\nu} :_{\mu} - \frac{1}{2} R g^{\mu\nu} = 0 = \kappa T^{\mu\nu} :_{\mu} , \]

(1.41)

where the left-hand side follows from the Bianchi identity. Thus the stress-energy tensor is a conserved "tensor current" and the associated symmetry is the general coordinate transformation under which EFE is invariant.

1.2.3 From Classical to Quantum General Relativity

In the section 1.1.3 we noted that the transformation \( (1.13) \) will cancel all the divergences in QED at any level of perturbation theory. We shall now examine this behavior more closely by inspecting the coupling constant of QED. As can be seen from the equations of motion \( (1.7) \), \( (1.8) \) the matter fields couple to the photon field through the electrical charge. The coupling constant depends thus on the electron charge \( e \), which can be expressed in natural units \( (c = \hbar = \epsilon_0 = 1) \) with the fine-structure constant \( \alpha \) as

\[ e = \sqrt{4\pi\alpha} . \]

(1.42)

Thus the coupling constant of QED is dimensionless.
When we calculate amplitudes for divergent loop-diagrams, such as in fig. 1.1.3, every loop will produce a factor of $e^2$ or $4\pi\alpha$ in the amplitude. The divergences will arise from these loops, so they will be proportional to the powers of the coupling constant, and hence, to $\alpha$. In the first order of perturbation theory, which corresponds to loop-diagrams with at most one loop, the divergences are proportional to $\alpha$. In the second order they are proportional to $\alpha^2$ and in the $n$:th order (diagrams with $n$ loops) they are proportional to $\alpha^n$.

Because the divergences in different levels of perturbation theory are proportional to different powers of the coupling constant, we might need different counterterms in each level of perturbation theory. Fortunately the coupling constant is dimensionless, and the divergences are proportional to the same combinations of the photon and electron fields in every order of the perturbation theory. This allows us to cancel all the divergences to the order $n$ of, say the interaction-vertex, with one renormalization constant, in this case $Z_1$. Hence all possible divergences can be removed in any level of perturbation theory with only three renormalization constants $Z_1$, $Z_2$ and $Z_3$. Transformation (1.13) associates these renormalization constants with the electron’s mass and charge, which we will have to determine experimentally. Thus the divergences are essentially removed by measuring the aforementioned two constants.

Now we shall compare this to the case of GR, in which the geometry of spacetime and the matter content are coupled through the stress-energy tensor. From Einstein field equations (1.39) we see that the coupling constant will be proportional to $\kappa$, which in natural units has the dimension of length squared. Thus the coupling constant of GR is not dimensionless, which will lead into troubles in the perturbation expansion.

As we already noted, in QED the divergences in the $n$:th level of perturbation theory are proportional to the $n$:th power of the coupling constant. In quantum GR the coupling constant has dimension of length squared and the divergences are hence proportional to the inverse powers of length. In fact the divergences will be proportional to linearly independent geometrical terms with units of length. Thus one cannot cancel them with a single renormalization constant, and we are forced to add new renormalization constants proportional to the new geometrical terms in each order of the perturbation theory, which corresponds to adding new higher order terms into the Lagrangian of general relativity (1.31).

In practice the new renormalization constants will introduce new physical constants which we must measure in order to make sense of the theory. But because each level of perturbation theory will introduce new constants we will eventually end up with infinitely many constants to be determined by the experiments. Clearly we can’t do an infinite number of measurements so in the end the theory won’t really predict anything. Hence we conclude that general relativity is not a renormalizable theory. There are also other concerns regarding the validity of the perturbation expansion and possible instabilities, but we will postpone the discussion of these into section 2.3. More thorough discussion about this topic is given for example in [21].
Chapter 2

Semiclassical Quantum Gravitation

After the discussion of the section 1.2.3 we turn our attention to semiclassical quantum gravitation. We shall begin this chapter by reviewing the fundamental concepts of such semiclassical theory. Then we shall tackle the problem of renormalization, and introduce the DeWitt-Schwinger expansion (DS-expansion). After identifying the divergent terms of the DS-expansion we use the counterterm renormalization from section 1.1.3 in order to render the effective Lagrangian finite.

The renormalized semiclassical theory leads to differential equations that are fourth order in the derivatives of the metric, which evokes a problem with stability. We review shortly the Ostrogradsky’s theorem in the context of general relativity and conclude that the semiclassical theory must contain unstable solutions. After this remark we will show that the instabilities are associated with unphysical solutions of the fourth order equations and that, in fact, all the physical solutions are stable.

2.1 Concepts of Semiclassical Quantum Gravitation

2.1.1 Quantum fields in a Background Spacetime

In section 1.2.3 we discussed about the renormalizability of quantum general relativity and concluded that it is not possible with the current theoretical framework. However, even if the full quantum theory of gravitation is out of our reach we might be able to formulate a semiclassical theory for gravitation. Such theory would then apply on the regime beyond the general relativity up to the Planck scale where the full quantum theory takes over. Most natural way to formulate semiclassical theory would be to consider small perturbations around a classical background metric. The perturbations would then come from the quantized matter field and the first order quantum gravitational interactions.

We shall begin the formulation of the semiclassical theory by considering
the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} . \] (2.1)

The matter-energy content of the spacetime enters these equations through the stress-energy tensor, which is obtained from classical tensor fields. Our first task is to replace the classical stress-energy tensor \( T_{\mu\nu} \) with a quantum mechanical expectation value \( \langle T_{\mu\nu} \rangle \) defined as

\[ \langle T_{\mu\nu} \rangle \equiv \frac{\langle \text{out}, 0 | T_{\mu\nu} | \text{in}, 0 \rangle}{\langle \text{out}, 0 | \text{in}, 0 \rangle} , \] (2.2)

where \( |\text{in}\rangle \) and \( \langle \text{out}| \) are the in and out vacuum states. Formally the expectation value is obtained from the quantum matter fields. However, in practice we usually go the other way around and calculate an effective action for the matter fields from the expectation value through the relation

\[ \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle . \] (2.3)

Let’s now examine more closely the modified field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa \langle T_{\mu\nu} \rangle . \] (2.4)

The expectation value on the right hand side contains all the quantum effects and since we are working with the first order perturbation theory, we should expand it to first order in the powers of \( \hbar \)

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \hbar \kappa \langle T_{\mu\nu}^{\text{geom}} \rangle = -\kappa \langle T_{\mu\nu}^{\text{cl}} \rangle + O(\hbar^2) . \] (2.5)

The first term on the right hand side is the classical stress-energy tensor that generates the background metric and is independent of the quantum state. The second term contains the first order corrections to the background. It contains two kind of terms, the first kind being purely geometrical and the second kind being dependent of the quantum state of the matter fields. Hence we shall split the second term into two parts

\[ \langle T_{\mu\nu}^{1} \rangle = \langle T_{\mu\nu}^{\text{geom}} \rangle + \langle T_{\mu\nu}^{\text{matter}} \rangle , \] (2.6)

and move the geometrical part onto the left hand side of (2.5)

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \hbar \kappa \langle T_{\mu\nu}^{\text{geom}} \rangle = -\kappa \langle T_{\mu\nu}^{\text{cl}} \rangle - \hbar \kappa \langle T_{\mu\nu}^{\text{matter}} \rangle + O(\hbar^2) . \] (2.7)

This is the framework for the semiclassical theory of gravitation we will be working with.

The new geometrical term in the left hand side of (2.7) is now the first order contribution of the quantum interactions to the classical Einstein-Hilbert Lagrangian. On dimensional grounds it will be fourth order in the derivatives of the metric and hence (2.7) is fourth order differential equation for the metric. Thus the quantum corrections change radically the character of the Einstein field equations by expanding the space of possible solutions as we will see in the section 2.3.
2.1.2 Coupling the Matter to Gravity

After the perturbative considerations of the previous section we shall now focus on the renormalization of effective action $S_{\text{eff}}$, which produces the expectation value $\langle T_{\mu \nu} \rangle$. We shall begin by writing out the classical action for scalar field coupled to gravitation. Then we note that the variation of this action with respect to the scalar field produces the equation of motion for the classical field. This equation of motion can then be converted into an expression for the quantum mechanical Green’s function.

The Green’s function in turn can be used to construct an asymptotic expansion for the effective semiclassical Lagrangian in the ultraviolet limit. This is done by using the DeWitt-Schwinger representation (DS-representation) with dimensional regularization and point-splitting. The first few leading terms of the asymptotic expansion will be divergent, and in the end we will remove the divergences of the effective Lagrangian by using the counterterm renormalization introduced in 1.1.3.

Let us consider the most simple example of scalar field coupled to the gravitation. The classical Lagrangian in four dimensions for such field is

$$L_{\text{cl}} = \frac{1}{2} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi \mathcal{R} \phi^2 .$$

(2.8)

The curved space manifests itself in this equation by two ways: firstly through the metric in the first term and secondly through the last term in which the gravitation couples to the scalar field.

The constant $\xi$ determines the type of the gravitation-matter coupling. Two important special cases are the minimal coupling $\xi = 0$ and the conformal coupling defined in $n$ spacetime dimensions as

$$\xi(n) = \frac{n - 2}{4(n - 1)} .$$

(2.9)

For two spacetime dimensions these couplings are equal as can be easily seen from (2.9).

The equation of motion for the scalar field can be derived by varying the classical action

$$S_{\text{cl}} = \int d^4x \sqrt{-g} L_{\text{cl}} ,$$

(2.10)

with respect to the scalar field

$$\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{cl}}}{\delta \phi} = g^{\alpha \beta} \nabla_\alpha \nabla_\beta \phi + m^2 \phi + \xi \mathcal{R} \phi = 0 .$$

(2.11)

Thus the classical field must satisfy the classical equation of motion

$$[\Box + m^2 + \xi \mathcal{R}] \phi = 0 .$$

(2.12)

where the square denotes the generalized d’Alembert operator

$$\Box \phi \equiv g^{\alpha \beta} \nabla_\alpha \nabla_\beta \phi .$$

(2.13)
The classical equation of motion (2.12) can be used to construct the quantized field operator for $\phi$ by finding the mode solutions. More importantly we can use (2.12) to find the Feynman Green’s function for the scalar field

$$
\left[ \Box_x + m^2 + \xi R(x) \right] G_F(x, x') = -\frac{\delta^n(x-x')}{\sqrt{-g(x)}},
$$

(2.14)

which we can use to construct the effective action for $\phi$. Note also, that we have moved to $n$-spacetime as we will need the $n$-dimensional Lagrangian for the purposes of dimensional regularization. Formally equation (2.14) has the solution

$$
G_F(x, x') = -\left[ \Box_x + m^2 + \xi R(x) \right]^{-1} \frac{\delta^n(x-x')}{\sqrt{-g(x)}}.
$$

(2.15)

Our next task is to use some formal manipulations in order to convert this abstract expression for the Green’s function into an expression for the effective quantum mechanical action.

### 2.1.3 Constructing the Effective Action from Green’s functions

In order to derive the effective action for the scalar field in terms of the Green’s function we need to consider the generating functional

$$
Z[J] = \langle 0, \text{out} | 0, \text{in} \rangle_J = \int \mathcal{D}[\phi] \exp \left\{ i S_m[\phi] + i \int d^n x J(x) \phi(x) \right\}.
$$

(2.16)

This functional gives the transition amplitude from the initial vacuum to the final vacuum in the presence of a particle producing source $J(x)$. By setting $J = 0$ and examining the variation of

$$
Z[0] \equiv \langle 0, \text{out} | 0, \text{in} \rangle_{J=0},
$$

(2.17)

the sourceless vacuum persistence amplitude, we see that

$$
\delta Z[0] = i \int \mathcal{D}[\phi] \delta S_m e^{iS_m[\phi]} = i \langle \text{out, 0} | \delta S_m | \text{in, 0} \rangle.
$$

(2.18)

Then according to (2.3) we have

$$
\frac{2}{\sqrt{-g}} \frac{\delta Z[0]}{\delta g^{\mu\nu}} = i \langle \text{out, 0} | T_{\mu\nu} | \text{in, 0} \rangle.
$$

(2.19)

If we now express $Z[0]$ with the effective action as

$$
Z[0] = e^{i S_{\text{eff}}},
$$

(2.20)

and use this expression in (2.19) we obtain

$$
ie^{i S_{\text{eff}}} \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = i \langle \text{out, 0} | T_{\mu\nu} | \text{in, 0} \rangle.
$$

(2.21)
By dividing with $Z[0] = e^{iS_{\text{eff}}}$, we obtain

$$\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{eff}}}{\delta g_{\mu\nu}} = \frac{\langle \text{out}, 0 | T_{\mu\nu} | \text{in}, 0 \rangle}{\langle \text{out}, 0 | 0, \text{in} \rangle} = \langle T_{\mu\nu} \rangle,$$

where the operator $G_F$ satisfies

$$G_F(x, x') = \langle x \left| G_F \right| x' \rangle,$$

in the vector space spanned by $|x\rangle$. The trace of an operator $M$ in this space is defined as

$$\text{tr} M \equiv \int d^n x \sqrt{-g} \langle x \left| M \right| x \rangle.$$

By inverting (2.20) and using the relation (2.23), we obtain

$$S_{\text{eff}} = -i \ln Z[0] = -\frac{1}{2} i \text{tr} \left[ \ln (-G_F) \right].$$

In order to make sense of the formal definition (2.26), we need to use some representation of $G_F$. We start by defining a two-point function $K_{xy}$ and its inverse $K_{xy}^{-1}$ as

$$K_{xy} \equiv K \frac{\delta^n(x - y)}{\sqrt{-g}},$$

$$K_{xy}^{-1} \equiv K^{-1} \frac{\delta^n(x - y)}{\sqrt{-g}},$$

where $K$ is

$$K = \Box_x + m^2 - i\varepsilon + \xi R(x).$$

These definitions satisfy

$$\int d^n y \sqrt{-g} K_{xy} K_{yz}^{-1} = \frac{\delta^n(x - z)}{\sqrt{-g}},$$

which, when compared to (2.14), implies

$$G_F(x, z) = -K_{xz}^{-1}.$$

Now we shall use a representation derived by DeWitt and Schwinger (and hence named DeWitt-Schwinger representation), which gives the operator relation

$$G_F = -K^{-1} = -i \int_0^\infty e^{-iK_s s} ds.$$
Next we make use of the identity
\[
\int_{\Lambda}^{\infty} e^{-ik(s)\Lambda^{-1}}is = -\text{Ei}(-i\Lambda K),
\]
where Ei is the exponential integral function, which has the expansion
\[
\text{Ei}(x) = \gamma + \ln(-x) + \mathcal{O}(x).
\]
By using the expansion (2.34) in (2.33) we get
\[
\int_{\Lambda}^{\infty} e^{-ik(s)\Lambda^{-1}}is = \ln(K) + \gamma + \ln(i\Lambda) + \mathcal{O}(\Lambda).
\]
After taking the limit \(\Lambda \to 0\) and discarding an infinite metric independent constant we obtain
\[
\ln(-G_F) = -\ln(K) = \int_{0}^{\infty} e^{-ik(s)\Lambda^{-1}}is = \int_{m^2}^{\infty} dm^2 \int_{0}^{\infty} e^{-ik}s is ds,
\]
where the integration over \(m^2\) brings down the extra power of \((is)^{-1}\). Thus in the DeWitt-Schwingert representation
\[
\langle x | \ln(-G_F^{DS}) | x' \rangle = -\int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x').
\]
Now the formal expression (2.26) can be cast into the form
\[
S_{\text{eff}} = \frac{1}{2} i \int d^n x \sqrt{-g} \lim_{x \to x'} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x').
\]
By interchanging the order of the integral over \(x\) and the limit we get
\[
S_{\text{eff}} = \frac{1}{2} i \int_{m^2}^{\infty} dm^2 \int d^n x \sqrt{-g} G_F^{DS}(x, x).
\]
This is the one-loop effective action, and the corresponding Lagrangian is
\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} i \lim_{x \to x'} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x').
\]
Thus we have reduced the evaluation of the effective Lagrangian to the calculation of the Feynman Green’s function in curved space.

### 2.2 Renormalizing the Semiclassical Theory

#### 2.2.1 The DeWitt-Schwingert Expansion

Now that we have a way to calculate the effective Lagrangian we may turn our attention to the renormalization of \(\mathcal{L}_{\text{eff}}\). The Feynman Green’s function needed in (2.40) can be calculated from the equation of motion through (2.14).
Because we want to examine the ultraviolet behavior of the Lagrangian, we may expand (2.14) near the point $x'$ in the Riemann normal coordinates

$$y^\mu = (x - x')^\mu .$$  (2.41)

In these coordinates expansion of the metric to fourth order is

$$g_{\mu \nu}(x) = \eta_{\mu \nu} + \frac{1}{3} R_{\mu \alpha \nu \beta} y^\alpha y^\beta - \frac{1}{6} R_{\mu \alpha \nu \beta \gamma} y^\alpha y^\beta y^\gamma + \left[ \frac{1}{20} R_{\mu \alpha \nu \beta \gamma \delta} + \frac{2}{45} R_{\mu \alpha \nu \beta \lambda} R^{\lambda}_{\gamma \rho \delta} \right] y^\alpha y^\beta y^\gamma y^\delta + \cdots$$  (2.42)

In order to simplify the calculations we introduce also the modified Feynman Green's function $\mathcal{G}_F$ and its Fourier transform defined as

$$\mathcal{G}_F(x, x') = (-g)^{1/4}(x) \mathcal{G}_F(x, x')$$  (2.43)

$$\mathcal{G}_F(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{-i \eta_{\mu \nu} k_{\mu} y^\nu} \mathcal{G}_F(k) .$$  (2.44)

When we expand (2.14) in the normal coordinates and use the Fourier transform of the modified Green's function, we can solve $\mathcal{G}_F$ by iteration to any order in the derivatives of the metric. The result to fourth order is

$$\mathcal{G}_F(k) = (k^2 - m^2)^{-1} - \left( \frac{1}{6} - \xi \right) R(k^2 - m^2)^{-2} + \frac{i}{2} \left( \frac{1}{6} - \xi \right) R_{\alpha \beta} \partial^\alpha (k^2 - m^2)^{2} - \frac{1}{3} a_{\alpha \beta} \partial^\alpha \partial^\beta (k^2 - m^2)^{-2} + \left[ \left( \frac{1}{6} - \xi \right)^2 R^2 + \frac{2}{3} a^\lambda \right] (k^2 - m^2)^{-3} ,$$  (2.45)

where the geometrical tensor $a_{\alpha \beta}$ is

$$a_{\mu \nu} \equiv \frac{1}{120} R_{\mu \nu} - \frac{1}{2} \left( \frac{1}{6} - \xi \right) R_{\mu \alpha \nu} - \frac{1}{40} R_{\mu \nu ; \alpha \beta} + \frac{1}{30} R_{\mu \alpha} R_{\nu \beta} + \frac{1}{60} R_{\mu \alpha ; \nu} R_{\beta \alpha} + \frac{1}{60} R_{\alpha \beta ; \mu} R_{\gamma \rho \delta} .$$  (2.46)

Now we use (2.44) to convert the momentum space expression (2.45) back to the coordinate space. The resulting Green's function is then

$$\mathcal{G}_F(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{-i \eta_{\mu \nu} k_{\mu} y^\nu} \left[ a_0(x, x') + a_1(x, x') \left( -\frac{\partial}{\partial m^2} \right) \right]$$

$$+ a_2(x, x') \left( \frac{\partial}{\partial m^2} \right)^2 \frac{1}{k^2 - m^2} ,$$  (2.47)

where we have defined

$$a_0(x, x') \equiv 1$$  (2.48)

$$a_1(x, x') \equiv \left( \frac{1}{6} - \xi \right) R - \frac{1}{2} \left( \frac{1}{6} - \xi \right) R_{\alpha \beta} y^\alpha y^\beta$$  (2.49)

$$a_2(x, x') \equiv \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2 + \frac{2}{3} a^\lambda .$$  (2.50)
The geometrical quantities in (2.49), (2.50) are all evaluated at the point \( x' \).

In order to simplify the expression (2.47) further we use the DeWitt-Schwinger integral representation

\[
(k^2 - m^2 + i\varepsilon)^{-1} = -i \int_0^\infty ds e^{i(k^2 - m^2 + i\varepsilon)} .
\]  (2.51)

After exchanging the \( d^nk \) and \( ds \) integration we obtain the simple result

\[
G_F(x, x') = -i \frac{1}{(4\pi)^{n/2}} \int_0^\infty ds (is)^{-n/2} e^{-im^2s + (\sigma/2is)} \left[ a_0 + a_1 is + a_2(is)^2 \right] ,
\]  (2.52)

where \( \sigma(x, x') = \frac{1}{2} y_\alpha y^\alpha \). Last thing to do is to use (2.43) which yields the DeWitt-Schwinger representation for the Feynman Green’s function

\[
G^{\text{DS}}_F(x, x') = -i \frac{\Delta(x, x')}{(4\pi)^{n/2}} \int_0^\infty ds (is)^{-n/2} e^{-im^2s + (\sigma/2is)} \left[ a_0 + a_1 is + a_2(is)^2 \right] ,
\]  (2.53)

where \( \Delta(x, x') \) is the van Vleck determinant

\[
\Delta(x, x') = -\det[\partial_\mu \partial_\nu g(x)g(x')]^{-\frac{1}{2}},
\]  (2.54)

which in the Riemann normal coordinates simplifies to \([-g(x)]^{-\frac{1}{2}}\).

Now we may substitute the DeWitt-Schwinger expansion (2.53) into the equation (2.40) in order to obtain the effective Lagrangian at the ultraviolet regime. The asymptotic expansion of the Lagrangian is then

\[
\mathcal{L}_\text{eff} = \frac{1}{2} \lim_{x \to x'} \frac{\Delta(x, x')}{(4\pi)^{n/2}} \int_0^\infty ds (is)^{-n/2 - 1} e^{-im^2s + (\sigma/2is)} \left[ a_0 + a_1 is + a_2(is)^2 \right] ,
\]  (2.55)

where we have already performed the \( dm^2 \) integral. If we now consider the spacetime dimension \( n \) to be an analytic variable we may take the coincidence limit \( x \to x' \), which yields

\[
\mathcal{L}_\text{eff} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} \int_0^\infty ds (is)^{-n/2 - 1} e^{-im^2s} \left[ a_0(x) + a_1(x)is + a_2(x)(is)^2 \right] ,
\]  (2.56)

where the local geometrical functions \( a_0, a_1 \) and \( a_2 \) are

\[
a_0(x) = 1 \\
a_1(x) = (\frac{1}{6} - \xi)R \\
a_2(x) = \frac{1}{2}(\frac{1}{6} - \xi)^2 R^2 + \frac{1}{3} a^\alpha \alpha .
\]  (2.57-2.59)

The \( ds \) integral can now be simplified into Gamma functions and we have

\[
\mathcal{L}_\text{eff} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} m^{n-1} \left[ m^4 \Gamma(-\frac{n}{2}) a_0(x) + m^2 \Gamma(1 - \frac{n}{2}) a_1(x) + \Gamma(2 - \frac{n}{2}) a_2(x) \right] .
\]  (2.60)
Next we have to fix the mass of the scalar field with a renormalization scale $\mu$ in order to retain the dimension of $L_{\text{eff}}$ as (length)$^{-4}$ when $n \neq 4$. Therefore we rewrite (2.60) as

$$L_{\text{eff}} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} \left(\frac{m}{\mu}\right)^{n-4} \left[ m^4 \Gamma(-\frac{n}{2})a_0(x) + m^2 \Gamma(1-\frac{n}{2})a_1(x) + \Gamma(2-\frac{n}{2})a_2(x) \right].$$

(2.61)

At this point we should note that the DeWitt-Schwinger expansion can be calculated to an arbitrary precision and the full DeWitt-Schwinger expansion for $L_{\text{eff}}$ is

$$L_{\text{eff}} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} \left(\frac{m}{\mu}\right)^{n-4} \sum_{k=0}^{\infty} a_k(x) m^{4-2k} \Gamma\left(k - \frac{n}{2}\right),$$

(2.62)

as given in chapter 6 of [3]. In four spacetime dimensions the first three terms, the ones we have been using in our calculation thus far, are divergent while the rest of the terms are finite. Hence it is sufficient for the purposes of renormalization to use the three first terms that diverge at $n \rightarrow 4$.

Last thing to do is to expand the Gamma functions and $(\frac{m}{\mu})^{n-4}$ in the effective Lagrangian (2.61) around $n = 4$ in order to get

$$L_{\text{eff}} = - (4\pi)^{-\frac{n}{2}} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[ \gamma + \ln \frac{m^2}{\mu^2} \right] \right\} \left( \frac{4m^4}{n(n-2)}a_0 - \frac{2m^2}{n-2}a_1 + a_2 \right)$$

(2.63)

Now it is clear that the asymptotic expansion of the effective Lagrangian diverges when $n \rightarrow 4$ because of the first term in the square brackets. The remaining task is now to remove the divergences multiplied by the geometric terms $a_0$, $a_1$ and $a_2$ by absorbing them into some suitable constants in the Lagrangian.

### 2.2.2 Renormalization in the Effective Lagrangian

For the purposes of renormalization we shall divide the effective Lagrangian into divergent and renormalized part. The total Lagrangian is then

$$L_0 = L_g + L_{\text{div}} + L_{\text{ren}},$$

(2.64)

where the lower index zero signifies that the Lagrangian is unrenormalized. In the previous section we found the divergent part of the effective Lagrangian in the asymptotic limit to be

$$L_{\text{div}} = - (4\pi)^{-\frac{n}{2}} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[ \gamma + \ln \frac{m^2}{\mu^2} \right] \right\} \left( \frac{4m^4}{n(n-2)}a_0 - \frac{2m^2}{n-2}a_1 + a_2 \right)$$

(2.65)
where the geometrical terms $a_0$, $a_1$ and $a_2$ can be expressed in the form

\begin{align*}
a_0 &= 1 \\
a_1 &= \left( \frac{1}{6} - \xi \right) R \\
a_2 &= \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{6} \left( \frac{1}{6} - \xi \right) \Box R + \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2 .
\end{align*}

The divergent part is purely geometrical and should thus be included in the gravitational Lagrangian

\begin{align*}
\mathcal{L}_g + \mathcal{L}_{\text{div}} &= \frac{1}{2\kappa} (R - 2\Lambda) + A a_0 + B a_1 + C a_2 \\
&= \frac{1}{2\kappa_0} \left\{ \left( 1 + \frac{1}{3} \kappa_0 B \right) R - 2(\Lambda_0 - \kappa_0 A) \right\} + C a_2 .
\end{align*}

The last term in (2.69) can not be absorbed into the gravitational Lagrangian and therefore we need to add new terms into $\mathcal{L}_g$

\begin{align*}
\mathcal{L}_g &= \frac{1}{2\kappa_0} (R - 2\Lambda_0) + \alpha_0 R^2 + \beta_0 R_{\alpha\beta} R^{\alpha\beta} + \gamma_0 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} ,
\end{align*}

where $\alpha_0$, $\beta_0$, $\gamma_0$ are new unrenormalized constants. With this new improved Lagrangian we obtain

\begin{align*}
\mathcal{L}_g + \mathcal{L}_{\text{div}} &= \frac{1}{2\kappa_0} \left\{ \left( 1 + \frac{1}{3} \kappa_0 B \right) R - 2(\Lambda_0 - \kappa_0 A) \right\} + \left( \alpha_0 + \frac{1}{12} C \right) R^2 \\
&+ \left( \beta_0 - \frac{1}{180} C \right) R_{\alpha\beta} R^{\alpha\beta} + \left( \gamma_0 + \frac{1}{180} C \right) R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} .
\end{align*}

where we have also removed the surface term $\Box R$, which variation will vanish anyway. From this we see that the constants in the Lagrangian will be renormalized as follows

\begin{align*}
\kappa &= \frac{\kappa_0}{1 + \frac{1}{3} \kappa_0 B} \\
\Lambda &= \Lambda_0 - \kappa_0 A \\
\alpha &= \alpha_0 + \frac{1}{12} C \\
\beta &= \beta_0 - \frac{1}{180} C \\
\gamma &= \gamma_0 + \frac{1}{180} C .
\end{align*}

In the special case of $n = 4$ the Gauss-Bonnet theorem states that

\begin{align*}
\int d^2 x \left[ -g(x) \right]^{\frac{1}{2}} \left( R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2 \right) ,
\end{align*}

(2.77)
is a topological invariant, which variation with respect to the metric will vanish. Hence in four dimensions only two of the coefficients $\alpha$, $\beta$ and $\gamma$ are independent, and we may choose $\gamma = 0$ for convenience. The renormalized gravitational Lagrangian is then

$$\mathcal{L}_g = \frac{1}{2\kappa}(R - 2\Lambda) + \alpha R^2 + \beta R_{\alpha\beta}R^{\alpha\beta}$$

(2.78)

where constants $\kappa$, $\Lambda$, $\alpha$ and $\beta$ are to be determined by experiments. The renormalized total Lagrangian is simply the sum of the renormalized gravitational Lagrangian and the renormalized part of the effective Lagrangian

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{\text{ren}}$$

(2.79)

The renormalized effective Lagrangian is by definition obtained from the original effective Lagrangian by subtracting the divergent part

$$\mathcal{L}_{\text{ren}} \equiv \mathcal{L}_{\text{eff}} - \mathcal{L}_{\text{div}}$$

(2.80)

The renormalized stress-tensor can then be calculated straight from the renormalized matter Lagrangian

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = -2\frac{\delta \mathcal{L}_{\text{ren}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{ren}}$$

(2.81)

### 2.3 Stability of the Semiclassical Solutions

#### 2.3.1 Ostrogradsky’s theorem

At this point we shall make a short intermezzo and review the Ostrogradsky’s theorem. Our treatment will follow loosely the one given by Woodard in [20] and [21]. First we shall note that the gravitational Lagrangian (2.78) can be expressed in the form

$$\mathcal{L}_g(g, \partial g, \partial^2 g) = A(\partial^2 g)(\partial^2 g) + B(\partial g, \partial^2 g)\partial g + C(g, \partial g, \partial^2 g)g + D$$

(2.82)

where $A, D$ are constants and $B, C$ are functions of the metric and its derivatives. We have suppressed the indices of the derivatives and the metric for simplicity. If $A \neq 0$, the Lagrangian is non-degenerate and we can use the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}_g}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial \mathcal{L}_g}{\partial (\partial_\alpha g^{\mu\nu})} + \partial_\alpha \partial_\beta \frac{\partial \mathcal{L}_g}{\partial (\partial_\alpha \partial_\beta g^{\mu\nu})} = 0$$

(2.83)

in order to solve the fourth derivative of the metric

$$\partial^4 g^{\mu\nu} = \mathcal{F}(g^{\mu\nu}, \partial g^{\mu\nu}, \partial^2 g^{\mu\nu})$$

(2.84)
Because the metric has only six independent components in $n = 4$, only six of these fourth order differential equations are independent. The solution to this set is of the form
\[ g^\mu_\nu(x) = Q(x, g^\mu_\nu_0, \partial g^\mu_\nu_0, \partial^2 g^\mu_\nu_0, \partial^3 g^\mu_\nu_0) \] (2.85)
where $g^\mu_\nu_0$, $\partial g^\mu_\nu_0$, $\partial^2 g^\mu_\nu_0$, and $\partial^3 g^\mu_\nu_0$ are the initial conditions or boundary conditions for the metric. Because only six of the components are independent we will need $6 \times (1 + 4 + 10 + 20) = 210$ boundary conditions, which means that there must be 210 canonical coordinates. We shall denote these coordinates with the tensors
\[ q^\mu_\nu \equiv g^\mu_\nu, \quad p^\kappa_\mu_\nu \equiv \partial L_g/\partial (\partial_\kappa g^\mu_\nu), \quad Q^\mu_\nu_\kappa \equiv \partial_\kappa g^\mu_\nu, \quad P^\kappa_\lambda_\mu_\nu \equiv \partial L_g/\partial (\partial_\kappa \partial_\lambda g^\mu_\nu) \] (2.86)

The non-degenerancy allows us to solve the second derivatives of the metric in terms of $q$, $Q$ and $P$ from the equation
\[ \frac{\partial L_g}{\partial (\partial_\kappa \partial_\lambda g^\mu_\nu)} \bigg|_V = P^\kappa_\lambda_\mu_\nu \] (2.87)
where the substitution is done as
\[ V = \begin{cases} g^\mu_\nu = q^\mu_\nu, \\ \partial_\kappa g^\mu_\nu = Q^\mu_\nu_\kappa, \\ \partial_\kappa \partial_\lambda g^\mu_\nu = a^\mu_\nu_\kappa_\lambda \end{cases} \] (2.88)

Note that (2.87) implies that $a^\mu_\nu_\kappa_\lambda$ is independent of the canonical coordinates $p^\kappa_\mu_\nu$ because $L_g$ is independent of $\partial^3 g$.

The Ostrogradsky’s Hamiltonian is obtained by Legendre transforming the Lagrangian
\[ H(q, Q, p, P) \equiv p^\kappa_\mu_\nu Q^\mu_\nu_\kappa + P^\kappa_\lambda_\mu_\nu a^\mu_\nu_\kappa_\lambda (q, Q, P) - L_g(q, Q, a(q, Q, P)) \] (2.89)
The evolution equations for the canonical coordinates are then those suggested by the notation
\[ \partial_\kappa q^\mu_\nu \equiv \frac{\partial H}{\partial p^\kappa_\mu_\nu}, \quad \partial_\kappa Q^\mu_\nu_\lambda \equiv \frac{\partial H}{\partial P^\kappa_\lambda_\mu_\nu}, \quad \partial_\alpha p^\kappa_\mu_\nu \equiv -\frac{\partial H}{\partial q^\mu_\nu}, \quad \partial_\alpha P^\kappa_\mu_\nu \equiv -\frac{\partial H}{\partial Q^\mu_\nu_\kappa} \] (2.90)
It can now easily be shown that $H$ corresponds to the energy up to a canonical transformation.

The problem with Ostrogradsky’s Hamiltonian is that it is linear in the canonical momenta $p^\kappa_\mu_\nu$. This linearity leads to arbitrarily large negative kinetic energies with a special sort of time dependence. The problematic term in the Hamiltonian is $p^\kappa_\mu_\nu Q^\mu_\nu_\kappa$, that can be made large by adjusting the third derivatives in $p$, which can be done while the dynamical variables $q^\mu_\nu$ are still small. This is especially bad in quantum field theory, where the linearity allows the vacuum to decay into particle pairs with equal positive and negative energy at arbitrary large energy scale. Hence the Ostrogradskian instability implies that the vacuum will decay instantly.
2.3.2 Stability of Physical Solutions

As we saw in the previous section, the Ostrogradsky’s theorem seems to indicate that the semiclassical quantum gravitation is unstable in a very fundamental way. However, closer inspection reveals that the instabilities are related to unphysical solutions of the fourth order differential equations. By removing these unphysical, badly behaving solutions the semi-classical theory can be rendered stable. The treatment given here was first proposed by Parker and Simon in [13].

Let us now examine more closely the semiclassical Einstein field equations of section 2.1.1 defined as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \hbar \langle T^{\text{geom}}_{\mu\nu} \rangle = -\kappa \langle T^{\text{cl}}_{\mu\nu} \rangle - \hbar \kappa \langle T^{\text{matter}}_{\mu\nu} \rangle + \mathcal{O}(\hbar^2) \]  

(2.91)

As we saw in section 2.2.2, the geometrical part of the stress-energy tensor expectation value \( \langle T^{\text{geom}}_{\mu\nu} \rangle \) contains terms that are fourth order in the derivatives of the metric. These terms are necessarily needed for the renormalization to work but they also change the character of the EFE, which become fourth order partial differential equations.

The equations being fourth order means that the solution space for the semiclassical EFE is much larger than that of the classical one and it will, in particular, contain unstable solutions that are not perturbatively expandable around some classical solution. In other words the semiclassical equation expanded to first order in \( \hbar \) can produce solutions comparable to arbitrary, and even negative powers of \( \hbar \). These kind of solutions are clearly not small corrections in powers of \( \hbar \) that we are looking for and should be discarded.

The non-perturbative solutions of the semiclassical EFE can be removed most efficiently by reducing the equations back to the second order. The reduction process will remove most of the unwanted solutions and retain all the physical solutions that are perturbatively expandable around some classical solution. However, some badly behaving solutions may still remain and they must be excluded by closer inspection.

The idea in the reduction of the order is to use the classical equation to bring the semiclassical equation back to the second order. This is most easily described by a simplified example. Consider the simple fourth order equation

\[ k \dddot{x}(t) + \dot{x}(t) + x(t) = 0 \]  

(2.92)

where \( k \) is a small constant, representing a perturbation resulting from semi-classical theory. Equation (2.92) is fourth order differential equation and will by definition contain unstable solutions. This can be seen by solving (2.92) exactly, which yields the solution

\[ x(t) = A \sin \omega_+ t + B \cos \omega_+ t + C \sin \omega_- t + D \cos \omega_- t \]  

(2.93)
where

\[
\omega_+ = \frac{\sqrt{1 + \sqrt{1 - 4k}}}{2k} = 1 + \frac{1}{2}k + \mathcal{O}(k) \tag{2.94}
\]

\[
\omega_- = \frac{\sqrt{1 - \sqrt{1 - 4k}}}{2k} = \frac{1}{\sqrt{k}} - \frac{1}{2}\sqrt{k} + \mathcal{O}(k) \tag{2.95}
\]

From the expansion of \(\omega_\pm\) around \(k = 0\) we can clearly see that some solutions are not perturbatively expandable at the "classical" limit \(k \to 0\).

At the limit \(k \to 0\) equation (2.92) simplifies to the "classical" equation

\[
\ddot{x}(t) + x(t) = 0 \tag{2.96}
\]

Differentiating this equation twice and multiplying by \(k\) we find that

\[
k\dddot{x} = -k\ddot{x} \tag{2.97}
\]

and by substituting (2.97) back to (2.92) we get the reduced semiclassical equation

\[
(1 - k)\dddot{x}(t) + x(t) = 0 \tag{2.98}
\]

The solutions to this equation are then

\[
x(t) = A\sin \omega t + B\cos \omega t \tag{2.99}
\]

where

\[
\omega = \frac{1}{\sqrt{1 - k}} = 1 + \frac{1}{2}k + \mathcal{O}(k) \tag{2.100}
\]

By examining the expansion of \(\omega\) we see that the reduction of order has removed the badly behaving solutions leaving only the physical solutions.

The calculation in the context of semiclassical quantum gravitation is much more complex but will follow the same principle. For the purposes of this thesis it is enough to know that such procedure exist and can be applied to de Sitter space, which is the most relevant spacetime for the problem of dark energy. Complete calculation for several common background metrics, including conformally flat metrics, is given in [13].

In the end we should note that while the reduction of order does remove the Ostrogradsky’s instability, there has been questions whether the reduction is physically justifiable [13]. There might be some unstable solutions that are physical but fail to show up in the observational data because of some other constraining conditions like long timescales. This subject is still under debate and might not be cleared before we have a complete quantum theory for gravitation.
Chapter 3

Trace Anomaly
and Local Degrees of Freedom

In this chapter we shall examine the trace anomaly, a quantum anomaly of the semiclassical gravitation. We begin by reviewing the conformal transformations, which are closely related to the trace anomaly. Then we derive the anomaly for conformally coupled scalar field in \( n \) dimensions. After that we will shortly review the Weyl cohomology and Wess-Zumino consistency, which help us to understand better the origins of the trace anomaly.

After the abstract considerations of section 3.1 we restrict our inspection into two-dimensions and derive the Wess-Zumino action for the trace anomaly. Then we construct the non-local anomalous action, which reproduces the trace anomaly in two-dimensions, and use an auxiliary scalar field in order to render the action local. It is then noted, that the auxiliary field adds a new local degree of freedom into the two-dimensional Lagrangian.

After considering the trace anomaly of conformal scalar field in two dimensions we turn our attention to the general trace anomaly in four dimensions. As in the two dimensional case we find the Wess-Zumino action and the anomalous action, which is then rendered local by introducing new auxiliary fields. In the end of this chapter we note that the auxiliary fields contain new degrees of freedom, which depend on the boundary conditions of the stress-energy tensor.

3.1 Trace anomaly and Weyl Cohomology
in \( n \)-dimensions

3.1.1 Conformal Transformations

Renormalization of the Lagrangian has some interesting, non-trivial consequences. For this thesis the most important one is the conformal anomaly or trace anomaly. As the name suggests, the anomaly relates closely to the conformal transformations which we shall review first.

The conformal transformations are defined as

\[
g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu} \equiv \bar{g}_{\mu\nu}(x),
\]  

(3.1)
where \( \sigma(x) \) is some function of \( x \). An expression which remains invariant under this transformation is called a conformal invariant or a Weyl invariant, and conformally transformed metrics \( \bar{g}_{\mu \nu} \) form the abelian Weyl group of the metric \( g_{\mu \nu} \). A metric that can be mapped from the Minkowski metric is said to be conformally flat

\[
g_{\mu \nu}(x) = e^{2\sigma(x)} \eta_{\mu \nu} \quad \text{conformally flat.} \tag{3.2}
\]

In the special case of two spacetime dimensions all the possible metrics are related by a conformal transformation, which allows us to construct any metric from the two-dimensional Minkowski metric. Hence all the two-dimensional metrics are conformally flat.

Let’s then consider the conformal transformation of the matter action \( S[g_{\mu \nu}(x)] \). By the definition of the functional integration

\[
S[\bar{g}_{\mu \nu}] = S[g_{\mu \nu}] + \int \frac{\delta S[g_{\mu \nu}(x)]}{\delta \bar{g}^{\sigma \sigma}(x)} \delta \bar{g}^{\rho \sigma}(x) d^n x . \tag{3.3}
\]

Now according to (3.1) we will have

\[
S[\bar{g}_{\mu \nu}] = S[g_{\mu \nu}] - \int \sqrt{-\bar{g}} T_{\rho}^{\rho}[g_{\mu \nu}] \delta \sigma d^n x , \tag{3.4}
\]

from which we get the trace of the stress-energy tensor

\[
T_{\rho}^{\rho}[g_{\mu \nu}] = - \frac{1}{\sqrt{-g}} \left. \frac{\delta S[\bar{g}_{\mu \nu}(x)]}{\delta \sigma} \right|_{\sigma=1} . \tag{3.5}
\]

Thus if the action is invariant under conformal transformations the classical stress-energy tensor will be traceless.

Finally let us note that in two and only two spacetime dimensions the classical action

\[
S = \int d^2 x \sqrt{-g} (\kappa R - 2\Lambda) \tag{3.6}
\]

is locally constrained, which means that it has no local degrees of freedom. This is because in two dimensions the metric has only two independent components, which are locally determined by the Einstein field equations.

### 3.1.2 Trace Anomaly Defined

Now that we have grasped some understanding of the conformal transformations we will start working with the actual anomaly. The simplest conformally invariant field is the conformally coupled massless scalar field, so we will begin by deriving the trace anomaly for such field. This leads us to examine the massless limit of the effective Lagrangian (2.61).

In the Lagrangian (2.61) the terms with \( a_0 \) and \( a_1 \) are multiplied by positive powers of \( m \) so they can clearly be set to zero. The \( a_2 \) term

\[
\frac{1}{2} \left[ \frac{-g(x)}{4\pi} \right]^{1/2} \left( \frac{m}{\mu} \right)^{n-4} \Gamma(2 - \frac{n}{2}) a_2(x) \tag{3.7}
\]
is somewhat trickier. First we need to express the geometrical quantity $a_2$ with two new local geometrical functions

\[ F(x) \equiv R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2 R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3} R^2 \]  
\[ G(x) \equiv R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2. \]

(3.8)

(3.9)

The geometrical function $a_2$ can then be expressed as

\[ a_2(x) = \alpha \left( F(x) - \frac{2}{3} \Box R \right) + \beta G(x) \],

(3.10)

where the coefficients are $\alpha = \frac{1}{120}$ and $\beta = -\frac{1}{360}$. We will now drop the surface term $\Box R$ which variation will vanish as before, and the $R^2$ term which is proportional to $(n - 4)^2$ with conformal coupling. The remaining terms can then be written as

\[ \frac{1}{2} \left[ -g(x) \right]^{\frac{3}{2}} \left( \frac{m}{\mu} \right)^{n-4} \Gamma(2 - \frac{n}{2}) [\alpha F(x) + \beta G(x)] \].

(3.11)

Next we use the identities

\[ 2 g^{\alpha\beta} \frac{\delta}{\sqrt{-g} \delta g^{\alpha\beta}} \int \sqrt{-g} F d^n x = -(n - 4) \left[ F - \frac{2}{3} \Box R \right] \]

(3.12)

\[ 2 g^{\alpha\beta} \frac{\delta}{\sqrt{-g} \delta g^{\alpha\beta}} \int \sqrt{-g} G d^n x = -(n - 4) G \]

(3.13)

in order to find the trace of the stress-energy tensor

\[ \langle T_{\alpha}^{\alpha} \rangle_{\text{div}} = \frac{2 g^{\alpha\beta} \delta S_{\delta \nu}}{\sqrt{-g} \delta g^{\alpha\beta}} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} \left( \frac{m}{\mu} \right)^{n-4} \Gamma(2 - \frac{n}{2}) \left[ \alpha \left( F - \frac{2}{3} \Box R \right) + \beta G \right] + O(n - 4) \].

(3.14)

Now we may expand the remaining terms in the powers of $(n - 4)$ and let $n \to 4$ in order to obtain the surprising result

\[ \langle T_{\alpha}^{\alpha} \rangle_{\text{div}} = \frac{1}{(4\pi)^2} \left[ \alpha \left( F - \frac{2}{3} \Box R \right) + \beta G \right] = \frac{a_2}{(4\pi)^2} \].

(3.15)

The divergent part of the stress-energy tensor has acquired a trace. Because the total action is conformally invariant, the total stress-energy tensor must be traceless

\[ \langle T_{\alpha}^{\alpha} \rangle = \langle T_{\alpha}^{\alpha} \rangle_{\text{div}} + \langle T_{\alpha}^{\alpha} \rangle_{\text{ren}} = 0 \]

(3.16)

and hence the renormalized stress-energy tensor will also acquire a trace

\[ \langle T_{\alpha}^{\alpha} \rangle_{\text{ren}} = -\frac{a_2(x)}{(4\pi)^2} \].

(3.17)
This is the trace anomaly. This result is easily extended to any even spacetime dimension \( n = 2k \) where

\[
\langle T_{\alpha}^{\alpha} \rangle_{\text{ren}} = -\frac{a_k(x)}{(4\pi)^k} .
\]  

For odd spacetime dimensions the trace anomaly vanishes identically. The expressions for higher spin fields are given in [5], pages 179-183.

We have worked here with massless field only for simplicity. Massive fields are not conformally invariant and hence will have an stress-energy tensor with non-zero trace, but there will still be an anomalous contribution to the trace. That said one might wonder whether the anomaly can be absorbed into the renormalization constants by making some finite adjustment.

There are no local counterterms which variation could cancel the entire anomaly. Of course if one would use more complicated, non-local action the anomaly could be removed, but there seems to be no reason to do that. It is also possible to remove the \( \Box R \) term from the anomaly by using the identity

\[
\frac{2}{\sqrt{-g}} g^{\alpha\beta} \frac{\delta}{\delta g^{\alpha\beta}} \int d^4x \sqrt{-g} R^2 = -12 \Box R ,
\]  

but this term will also break the conformal invariance of \( \mathcal{L}_{\text{eff}} \). That said, the anomaly itself is finite and it can very well be considered as a quantum correction to the classical trace of the stress-energy tensor, just as was with the breaking of the axial current conservation discussed in section 1.1.4 as there is no theoretical reason to remove it.

### 3.1.3 Weyl Cohomology

In order to discuss the trace anomaly in detail we will need some concepts of Weyl cohomology. First we shall introduce the finite difference operator \( \Delta_\sigma \), which acts on scalar functionals as

\[
\Delta_\sigma \circ S[\bar{g}] \equiv (\Delta S)_\sigma[\bar{g}] \equiv S[\epsilon^{2\sigma}\bar{g}] - S[\bar{g}] .
\]  

By definition, the Weyl invariant functionals will then satisfy

\[
\Delta_\sigma \circ S_{\text{inv}} = 0 .
\]  

We also define the action for a one-form as

\[
\Delta_{\sigma_2} \circ \Gamma[\bar{g}; \sigma_1] \equiv (\Delta \Gamma)_{\sigma_1}[\bar{g}; \sigma_2] - (\Delta \Gamma)_{\sigma_2}[\bar{g}; \sigma_1] ,
\]  

where \( \sigma_1 \) and \( \sigma_2 \) are two arbitrary Weyl transformations. This process can be repeated iteratively to obtain the action of \( \Delta_\sigma \) for a \( k \)-form.

Consider next the one-form

\[
\Gamma[\bar{g}; \sigma_1] = \Delta_{\sigma_1} S[\bar{g}] .
\]
By using (3.22) we obtain the identity
\[(\Delta_{\sigma_1} \circ \Delta_{\sigma_2}) \circ S[\bar{g}] = \Delta_{\sigma_1} \circ (\Delta_{\sigma_2} \circ S[\bar{g}]) = 0 \quad . \tag{3.24}\]

The Weyl cohomology can then be defined by closed and exact one-forms. A closed one-form \(\Gamma[\bar{g}; \sigma_1]\) satisfies
\[\Delta_{\sigma_2} \circ \Gamma[\bar{g}; \sigma_1] = 0 \quad \text{closed,} \tag{3.25}\]
and an exact one-form can be written as finite difference of some local, single-valued scalar functional
\[\Gamma[\bar{g}; \sigma_1] = \Delta_{\sigma_1} \circ S_{\text{local}}[\bar{g}] \quad \text{exact.} \tag{3.26}\]

By definition, every exact one-form is closed, but not every closed one-form needs to be exact. The trivial cohomology of the Weyl group is the group of exact one-forms, while the non-trivial cohomology is the group of closed, non-exact one-forms. Thus the one-forms of the non-trivial cohomology cannot be written in terms of some local functional.

### 3.1.4 Wess-Zumino consistency

Next we shall consider the Wess-Zumino (WZ) action, which satisfies the Wess-Zumino integrability condition for the abelian Weyl group
\[\frac{\delta^2 S}{\delta \sigma_1 \delta \sigma_2} - \frac{\delta^2 S}{\delta \sigma_2 \delta \sigma_1} = 0 \quad . \tag{3.27}\]

Let \(\Gamma_{WZ}[\bar{g}; \sigma]\) now be the one-form which variation with respect to \(\sigma\) generates the trace anomaly in 2\(k\)-dimensions, that is
\[\frac{\delta}{\delta \sigma} \Gamma_{WZ}[\bar{g}; \sigma] = -\sqrt{-g} \frac{a_k(x)}{(4\pi)^k} \quad . \tag{3.28}\]

If we now require that \(\Gamma_{WZ}\) is closed
\[\Delta_{\sigma_2} \circ \Gamma_{WZ}[\bar{g}; \sigma_1] = 0 \quad , \tag{3.29}\]
it will automatically satisfy the Wess-Zumino consistency condition (3.27) for finite shifts \(\Delta_{\sigma}\).

Our next step is to combine this information with the principles of dimensional regularization. In even spacetime dimensions we use local curvature invariants near the physical dimension as counterterms in order to cancel divergences represented by simple \(n-2k\) poles. These curvature invariants are not in general Weyl invariant.

The terms that are not Weyl invariant will contain simple pole in their Weyl shifts at \(n=2k\) and will generate exact one-forms of the trivial cohomology. These one-forms depend on the renormalization procedure and can be removed by finite shifts in the renormalization constants.
However, it may happen that some combination of the curvature invariants becomes a Weyl invariant in the physical dimension. These \( n \)-dimensional counterterms must then contain at least one \( n - 2k \) factor in their Weyl shift near \( n = 2k \), so that there exists the limit

\[
\Gamma[\bar{g}; \sigma] = \lim_{n\to 2k} \frac{\Delta_\sigma \circ S_n[\bar{g}]}{n - 2k} = \lim_{n\to 2k} \frac{S_n[e^{2\sigma} \bar{g}] - S_n[\bar{g}]}{n - 2k} .
\]  

(3.30)

When this limit exists, the resulting functional will automatically satisfy the WZ consistency condition for finite Weyl shifts. However, after taking the limit \( n \to 2k \), the effective action \( \Gamma[\bar{g}; \sigma] \) can no longer be written as the Weyl shift of a local action, and will hence belong to the non-trivial cohomology of the physical dimension \( n = 2k \). Thus the non-trivial cohomology of the Weyl group generates the trace anomaly.

### 3.2 The Effective Anomalous Action in Two Dimensions

#### 3.2.1 Wess-Zumino Action

Now we are all set up to find the exact form of the WZ-action. Before going into four dimensions we shall illustrate the general approach in two dimensions, where the calculations are greatly simplified and the physical behind them are seen more easily.

Near \( n = 2 \) dimensions the only possible curvature counterterm is the Ricci scalar. Hence we will consider the \( n \)-dimensional counterterm action

\[
S_n[g] = -\int d^n x \sqrt{-g} R ,
\]

(3.31)

and the corresponding WZ-action

\[
\Gamma_{WZ}[\bar{g}; \sigma] = -\lim_{n\to 2} \int d^n x \sqrt{-g} R - \int d^n x \sqrt{-\bar{g}} \bar{R} ,
\]

(3.32)

where \( R = R[g] \) is the Ricci scalar in \( n \) dimensions evaluated on the metric \( g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu} \) and \( \bar{R} = R[\bar{g}] \). In \( n \) dimensions the Weyl transformation of the Ricci scalar is

\[
\sqrt{-g} R = \sqrt{-\bar{g}} e^{(n-2)\sigma} \left[ \bar{R} - 2(n-1)\Box \sigma - (n-1)(n-2)\sigma_{,\alpha}\sigma^{,\alpha} \right] ,
\]

(3.33)

where all covariant derivatives and contractions are performed with the metric \( \bar{g}_{\mu\nu} \).

Expanding (3.33) to the first order in \( (n-2) \), substituting the result into (3.32) and taking the limit we find

\[
\Gamma_{WZ}[\bar{g}; \sigma] = -\int d^2 x \sqrt{-\bar{g}} \left[ \sigma \bar{R} - 2\sigma \Box \sigma - \sigma_{,\alpha}\sigma^{,\alpha} \right] .
\]

(3.34)
where we have ignored any vanishing surface terms. Integrating by parts the last term and discarding again a surface term we get, up to a multiplicative constant, the Polyakov action

$$\Gamma_{WZ}[\bar{g};\sigma] = -\int d^2x \sqrt{-\bar{g}} \left[ -\sigma \Box \sigma + \sigma R \right] \ .$$  \hspace{1cm} (3.35)

This action was first found by functionally integrating the trace anomaly with respect to $\sigma$ in two dimensions by A. M. Polyakov in [15]. Indeed by taking the variation of $\Gamma_{WZ}$ with respect to $\sigma$ we obtain

$$\frac{\delta \Gamma_{WZ}}{\delta \sigma} = -\sqrt{-\bar{g}}(\bar{R} - 2\Box \sigma) = -\sqrt{-g}R \ .$$  \hspace{1cm} (3.36)

Thus the Wess-Zumino consistent action producing the trace anomaly in two dimensions is

$$\Gamma_{WZ}[\bar{g};\sigma] = -\frac{1}{24\pi} \int d^2x \sqrt{-\bar{g}} \left[ -\sigma \Box \sigma + \sigma R \right] \ .$$  \hspace{1cm} (3.37)

### 3.2.2 Non-Local Anomalous Action

Now that we have found the WZ consistent action for the trace anomaly our next task is to form a non-local anomalous action, which Weyl shift corresponds to $\Gamma_{WZ}$

$$\Gamma_{WZ}[\bar{g};\sigma] \equiv \Delta_\sigma \circ S_{\text{anom}} [\bar{g}] = S_{\text{anom}} [g] - S_{\text{anom}} [\bar{g}] \ .$$  \hspace{1cm} (3.38)

The anomalous action is found by formally solving $\sigma$ from the transformation law

$$\sqrt{-g}R = \sqrt{-\bar{g}}(\bar{R} - 2\Box \sigma) \ .$$  \hspace{1cm} (3.39)

In order to do this we need to solve the Green’s function inverse of the second order differential operator $\Box$, which is the unique Weyl invariant differential operator in two dimensions

$$\sqrt{-\bar{g}}\Box = \sqrt{-g}\Box \ .$$  \hspace{1cm} (3.40)

The inversion of a differential operator is necessarily a non-local operation and results in a non-local action as we shall soon see.

The inversion can be done formally by solving the Green’s function inverse $D_2(x, x')$ of $\Box$ from the equation

$$\Box D_2(x, x') = -\frac{\delta(x, x')}{\sqrt{-g}} \ .$$  \hspace{1cm} (3.41)

By using this relation in (3.39) and solving for $\sigma$ we obtain the formal expression

$$\sigma(x) = \frac{1}{2} \int d^2x' D_2(x, x') \left[ \sqrt{-g'}R' - \sqrt{-\bar{g}'}\bar{R}' \right] \ .$$  \hspace{1cm} (3.42)
where the primes mean that the quantities are calculated for \(x',\) for example
\[ R' = R(x'). \]

With the solution for \(\sigma\) we can cast the WZ-action of (3.37) into the form
\[
\Gamma_{WZ}[\bar{g}; \sigma] = \frac{1}{96\pi} \int d^2x \sqrt{-g} \int d^2x' \sqrt{-g'} R D_2(x, x') R' 
- \frac{1}{96\pi} \int d^2x \sqrt{-\bar{g}} \int d^2x' \sqrt{-\bar{g}'} \bar{R} D_2(x, x') \bar{R}'.
\]  
(3.43)

The anomalous action can now easily be read from (3.43) and is
\[
S_{\text{anom}}[g] = \frac{1}{96\pi} \int d^2x \sqrt{-g} \int d^2x' \sqrt{-g'} R D_2(x, x') R'.
\]  
(3.44)

This action is clearly non-local, which signifies the fact that the trace anomaly is a consequence of the non-trivial Weyl cohomology.

### 3.2.3 The Localized Anomalous Action

Now that we have found the non-local action (3.44), we would like to convert it into a local form in order to study more closely the physics it describes. This can be done by introducing a new classical scalar field \(\varphi\), which will enclose all the non-local behavior. This auxiliary field is defined as
\[
-\Box \varphi = R.
\]  
(3.45)

The non-local origin of \(\varphi\) is easily seen when we use the inversion of \(\Box\) in (3.45) resulting in
\[
\varphi(x) = \int d^2x' \sqrt{-g} D_2(x, x') R'.
\]  
(3.46)

The anomalous action (3.44) can be expressed in terms of the auxiliary field as
\[
S_{\text{anom}} = \frac{1}{96\pi} \int d^2x \sqrt{-g} \left[ g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi - 2R \varphi \right].
\]  
(3.47)

It is easy to verify that the variation of (3.47) with respect to the auxiliary field leads to the equation of motion (3.45), and that when the formal solution (3.46) is substituted into (3.47) the non-local form of the action is restored.

At this point it is worth noting, that the auxiliary field acts as a new local degree of freedom in the action. In the beginning of this chapter we noted that the classical action for the gravity is locally constrained in two dimensions. This changes in the semiclassical theory, where the trace anomaly allows us to add the scalar auxiliary field to the action. This new field acts then as a new scalar degree of freedom, which is sensitive to the boundary conditions set upon the auxiliary field. In particular we have the possibility to add homogeneous solutions of (3.45) into the auxiliary field.
Finally we consider the anomalous stress-energy tensor $T_{\mu\nu}^{\text{anom}}$, which is obtained by varying the anomalous action (3.47) with respect to the metric

$$T_{\mu\nu}^{\text{anom}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{anom}}}{\delta g^{\mu\nu}} = \frac{1}{24\pi} \left[ -\varphi_{;\mu\nu} + g_{\mu\nu} \Box \varphi - \frac{1}{2} \varphi_{;\mu} \varphi_{;\mu} + \frac{1}{4} g_{\mu\nu} \varphi \varphi_{;\alpha} \varphi^{;\alpha} \right].$$

(3.48)

It is clear from (3.48) that the exact form of the anomalous energy tensor will depend on the boundary conditions imposed on the auxiliary field. By taking the trace of $T_{\mu\nu}^{\text{anom}}$ we get the standard form of the trace anomaly

$$g^{\alpha\beta} T_{\alpha\beta}^{\text{anom}} = \frac{1}{24\pi} \Box \varphi = -\frac{1}{24\pi} R.$$  

(3.49)

Thus the circle is finally complete.

### 3.3 Local Auxiliary Fields in Four Dimensions

#### 3.3.1 Trace Anomaly in Four Dimensions

We shall now explore the trace anomaly in four dimensions with multiple massless fields. Our treatment will follow the one given by Antoniadis, Mazur and Mottola in [3]. In four dimensions the trace of the stress-energy tensor takes the form

$$\langle T_{\alpha}^{\alpha} \rangle = bF + b' \left( G - \frac{2}{3} \Box R \right) + b'' \Box R + \sum_i \beta_i H_i,$$

(3.50)

where $F$ and $G$ are defined by (3.3), (3.9) and $\beta_i H_i$ are additional terms that may appear, if the fields couple to additional long range gauge fields. For example in the case of massless fermions coupled to an electromagnetic field one has $H = F_{\alpha\beta} F_{\alpha\beta}$. The coefficients $b$, $b'$ and $b''$ are dimensionless numbers multiplied by $\hbar$.

As before, the third in (3.50) can be removed by adding local terms into the Lagrangian which means its coefficient $b''$ depends on the renormalization scheme. Hence this term is of no interest to us, unlike the first two terms which can not be removed by adding local terms. The coefficients of these two terms are

$$b = -\frac{1}{120(4\pi)^2} \left( N_S + 6N_F + 12N_V \right),$$

(3.51)

$$b' = \frac{1}{360(4\pi)^2} \left( N_S + \frac{11}{2} N_F + 62N_V \right),$$

(3.52)

where $N_S$, $N_F$, $N_V$ are the number of fields with spin 0, $\frac{1}{2}$, 1 respectively.
3.3.2 Wess-Zumino Action and the Anomalous Action

We shall now examine the anomalous contribution to the trace defined as

\[
(T_{\text{anom}})_\alpha = bF + b' \left( G - \frac{2}{3} \Box R \right). \tag{3.53}
\]

As in the two-dimensional case, we want to find an effective Wess-Zumino action corresponding to this trace

\[
\frac{\delta W[g; \sigma]}{\delta \sigma} = \sqrt{-g}(T_{\text{anom}})_\alpha. \tag{3.54}
\]

In conformal transformations \( F \) and \( G \) will transform as

\[
\sqrt{-g}F = \sqrt{-\bar{g}} \bar{F} \tag{3.55}
\]

\[
\sqrt{-g} \left( G - \frac{2}{3} \Box R \right) = \sqrt{-\bar{g}} \left( \bar{G} - \frac{2}{3} \bar{\Box} \bar{R} \right) + 4\sqrt{-\bar{g}} \Delta_4 \sigma. \tag{3.56}
\]

The fourth order differential operator \( \Delta_4 \) used in (3.56) is defined as

\[
\Delta_4 \equiv \Box^2 + 2 R^{\alpha \beta} \nabla_\alpha \nabla_\beta - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\alpha R) \nabla_\alpha, \tag{3.57}
\]

which makes it the unique fourth order conformally covariant scalar differential operator

\[
\sqrt{-g} \Delta_4 = \sqrt{-\bar{g}} \bar{\Delta}_4. \tag{3.58}
\]

By inserting (3.55), (3.56) into the functional equation (3.54) we see that the Wess-Zumino action \( W[g; \sigma] \) is quadratic in \( \sigma \). The solution is then

\[
W[g; \sigma] = b \int d^4x \sqrt{-g} F \sigma + b' \int d^4x \sqrt{-g} \left\{ \left( G - \frac{2}{3} \Box \right) \sigma + 2 \sigma \Delta_4 \sigma \right\}. \tag{3.59}
\]

Now by formally solving \( \sigma \) from (3.56) and substituting the result into (3.59) we obtain

\[
W[g; \sigma] = S_{\text{anom}}[g = e^{2\sigma \bar{g}}] - S_{\text{anom}}[\bar{g}], \tag{3.60}
\]

where the non-local anomalous action is

\[
S_{\text{anom}}[g] = \frac{1}{2} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} \left( G - \frac{2}{3} \Box R \right) \nabla^\alpha R \nabla_\alpha \Delta_4^{-1}(x, x') \left[ bF' + b' \left( \frac{G'}{2} - \frac{\Box R'}{3} \right) \right]. \tag{3.61}
\]

Here \( \Delta_4^{-1}(x, x') \) denotes the Green’s function inverse of the fourth order operator defined in (3.57), and the primed geometrical quantities are evaluated in \( x' \). If there are any additional Weyl invariant terms in the anomaly, they should be included in the \( S_{\text{anom}} \) by replacing \( bF \rightarrow bF + \sum_i \beta_i H_i \) in the last square bracket.
3.3.3 Anomalous Local Degrees of Freedom

Next step is to render the non-local action $S_{\text{anom}}$ into a local form with additional degrees of freedom as described by Mottola in [11]. As in the two-dimensional case, this is done by introducing two new classical fields defined by

$$\Delta_4 \varphi = \frac{1}{2} G - \frac{1}{3} \Box R$$

(3.62)

$$\Delta_4 \psi = \frac{1}{2} F$$

(3.63)

The anomalous action can then be written as

$$S_{\text{anom}}[g; \varphi, \psi] = \frac{b}{2} \int d^4x \sqrt{-g} \left\{ -2 \varphi \Delta_4 \psi + F \varphi + \left( G - \frac{2}{3} \Box R \right) \psi \right\} + \frac{b'}{2} \int d^4x \sqrt{-g} \left\{ -\varphi \Delta_4 \varphi + \left( G - \frac{2}{3} \Box R \right) \varphi \right\}. \quad (3.64)$$

The non-local behavior of $S_{\text{anom}}$ is now encoded in the classical auxiliary fields $\varphi$ and $\psi$. The auxiliary fields depend on macroscopic border conditions and provide two new scalar degrees of freedom into the anomalous action.

The anomalous action (3.64) gives rise to the anomalous energy tensor

$$T_{\text{anom}}^{\mu \nu} = b F_{\mu \nu} + b' G_{\mu \nu}, \quad (3.65)$$

where $F_{\mu \nu}$ and $G_{\mu \nu}$ are given in [10] (equations 5.2 and 5.3 respectively). These tensors are locally conserved and they have the traces

$$F_{\alpha}^{\alpha} = 2 \Delta_4 \psi = F$$

(3.66)

$$G_{\alpha}^{\alpha} = 2 \Delta_4 \varphi = G - \frac{2}{3} \Box R$$

(3.67)

which generate the anomalous trace (3.53).
Chapter 4
Dynamical Vacuum Energy in de Sitter Space

In this chapter we will use the general theory built in the previous chapter to study the vacuum energy in de Sitter space. At first we shall review shortly the Friedman-Robertson-Walker -model (FRW-model) for the universe. Then we will consider the classical perturbation of FRW-universe, first in general and then for a scalar field.

After reviewing the FRW-model we will consider vacuum energy that arises from the trace anomaly. At first we will give two simple examples of trace anomaly induced vacuum energy. The first one is a simple cosmological term obtained in Minkowski space while the second one is the Starobinsky’s inflation model. After these examples we will consider the cosmological term induced by the trace anomaly in de Sitter space. Then we consider the perturbations of de Sitter space in the presence of the trace anomaly. In the end we note that the trace anomaly with classical auxiliary fields can’t produce the observed cosmological constant.

After the considerations of section 4.2 we will finally quantize the auxiliary fields and examine the dynamical effects of the trace anomaly. At first we will calculate the anomalous scaling dimension for the dimensionless cosmological constant and deduce, that it will flow into zero at large distances. Then we will examine the linear backreaction in de Sitter space when both the metric and the quantum state of the auxiliary fields are allowed to vary freely. We then find cosmological horizon modes for the auxiliary fields, which we examine more closely in the last section.

4.1 Friedman-Robertson-Walker universe

4.1.1 The Friedman-Robertson-Walker -model of the Universe

Before going to the cosmological applications of the trace anomaly we shall review the Friedman-Robertson-Walker (FRW) model of the universe. The FRW metric is the unique spatially isotropic and homogeneous solution of the
Einstein field equations with a constant scalar curvature. It is also assumed
that the space part of the metric may depend of the time coordinate. The
metric of the FRW universe can then be expressed in reduced-circumference
polar coordinates as
\[ ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right] , \quad (4.1) \]
where \( a(t) \) is the scale factor of the universe and \( K \) is one of \(+1, 0, -1\). Different
values of \( K \) correspond to elliptical space, Euclidean space and hyperbolic
space respectively.

The stress-energy tensor of the FRW universe is also assumed to be isotropic
and homogeneous, which leads us to
\[ T_{\mu\nu} = \text{diag}(\rho, -p, -p, -p) \quad , \quad (4.2) \]
where the energy density \( \rho \) and pressure \( p \) depend only on the time coordinate.
The time-time component of the Einstein field equations is then
\[ \frac{\dot{a}^2 + K}{a^2} = \frac{\kappa \rho + \Lambda}{3} \quad , \quad (4.3) \]
while the trace of EFE gives the equation
\[ \frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) + \frac{\Lambda}{3} \quad . \quad (4.4) \]
These two equations are the Friedman equations that give the time evolution
of the FRW universe. By using the first equation in the second one the Friedman
equations can be converted into an evolution equation for the energy density
\[ \dot{\rho} = -3H(\rho + p) \quad , \quad (4.5) \]
where \( H \) is the Hubble parameter
\[ H \equiv \frac{\dot{a}}{a} \quad . \quad (4.6) \]

The Friedman equations can be solved exactly for a perfect fluid with equation
of state
\[ p = w\rho \quad , \quad (4.7) \]
where \( w \) is some constant. The dependence between \( \rho \) and \( a \) can then be easily
solved from (4.5) and the result is
\[ \rho = \frac{\rho_0}{a_0} a^{-3(1+w)} \quad . \quad (4.8) \]
The solution for \( a(t) \) can then be obtained by substituting (4.8) into (4.3).
The resulting equation is
\[ \int_{a_0}^{a(t)} \left( \frac{\kappa \rho_0}{3 a_0} a^{-3(1+w)+2} + \frac{\Lambda}{3} a^2 - K \right)^{-\frac{1}{2}} da = t \quad , \quad (4.9) \]
from which $a(t)$ needs to be solved.

In general it is not possible to solve (4.9) analytically. Fortunately in the case of flat space $K = 0$, that we are interested in, the integral on the left-hand side of (4.9) can be solved in a closed form. If we also assume that the space is dominated by one form of matter or energy at a time we may set the cosmological constant to zero, which simplifies the calculations further. The scale factor $a(t)$ for a matter with equation of state $p = w\rho$ is then simply

$$a(t) = a_0 t^{\frac{2}{3(w+1)}} ,$$

(4.10)

where $a_0$ is determined from initial conditions.

Cosmologically interesting cases are the ordinary matter or "dust" with $w = 0$ and radiation with $w = 1/3$. Note also, that the solution (4.10) is not defined for $w = -1$, which corresponds to the vacuum energy represented by the cosmological constant. In a universe dominated by a cosmological constant the scale factor grows exponentially as

$$a(t) = a_0 e^{Ht} .$$

(4.11)

The particular solution of the Einstein field equations corresponding to (4.11) is de Sitter space, on which we will focus in the sections 4.2.1 and 4.3. De Sitter space is the closest approximation for the current state of the universe as 75 percent of the matter-energy content of our universe is calculated to be vacuum energy.

It is also useful to note that for a flat FRW-space the metric can be expressed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} ,$$

(4.16)

where $\tau$ is the conformal time related to the coordinate time $t$ as

$$d\tau = \frac{1}{a(t)} dt .$$

(4.13)

The scale factor can then be related with a conformal transformation as

$$a(t) = e^{\sigma(t)} .$$

(4.14)

By differentiating (4.14) with respect to the coordinate time and dividing with $a(t)$ we obtain another useful relation between $\sigma$ and the Hubble constant

$$H(t) = \frac{\dot{a}}{a} = \dot{\sigma}(t) .$$

(4.15)

### 4.1.2 Classical Perturbations of the FRW-model

Next we will discuss shortly the linear perturbations of the FRW metric in classical general relativity. Our approach will follow mostly the one given by Mukhanov, Feldman and Brandenberger in [12]. Let’s now consider the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} ,$$

(4.16)
where $\bar{g}_{\mu\nu}$ is the background FRW metric and $h_{\mu\nu}$ is a small perturbation around the exact solution. The most general perturbation is of the form

$$h_{\mu\nu} = \left( \begin{array}{cc} 2\varphi & 0 \\ -B_{i}^{j} - S_{i} & 2(\psi \bar{g}_{ij} - E_{(ij)} + F_{(ij)} + H_{ij}) \end{array} \right),$$

(4.17)

where $\varphi$, $\psi$, $B$ and $E$ are scalars, $S_{i}$ and $F_{i}$ are vectors and $H_{ij}$ is a tensor. The vectors satisfy the constraints

$$S_{i}^{j; i} = F_{i}^{j; i} = 0,$$

(4.18)

while the tensor must be symmetric and satisfy

$$H_{i}^{j; i} = H_{ij}^{j; i} = 0.$$

(4.19)

In the linear approximation scalar, vector and tensor perturbations evolve independently. Of these three types of perturbation only the scalar perturbations may lead to inhomogeneities in an expanding universe. Hence we will consider only the scalar perturbations from here on.

The scalar perturbations leave the metric with four degrees of freedom, two of which are fixed when we choose a gauge. One particularly interesting gauge is the conformal-Newtonian gauge in which $E = B = 0$ and the perturbation becomes diagonal

$$h_{\mu\nu} = \left( \begin{array}{cc} 2\varphi & 0 \\ 0 & 2\psi \bar{g}_{ij} \end{array} \right).$$

(4.20)

With a conformally flat background metric $\varphi = \psi$, and the perturbation is represented by a single degree of freedom

$$h_{\mu\nu} = 2\varphi \left( \begin{array}{cc} 1 & 0 \\ 0 & \bar{g}_{ij} \end{array} \right).$$

(4.21)

This property holds for any gauge and implies that while considering perturbations of a conformally flat metric it is enough to consider the time-time component of the Einstein field equations.

In the linear approximation the Einstein field equations can be written separately for the background metric and the perturbation as

$$\bar{R}^{\mu}_{\nu} - \frac{1}{2} \bar{R} \delta^{\mu}_{\nu} + \Lambda \delta^{\mu}_{\nu} = -\kappa \bar{T}^{\mu}_{\nu},$$

(4.22)

$$\delta R^{\mu}_{\nu} - \frac{1}{2} \delta R \delta^{\mu}_{\nu} = -\kappa \delta T^{\mu}_{\nu},$$

(4.23)

where $\delta R_{\mu\nu}$, $\delta R$ and $\delta T_{\mu\nu}$ are the linear perturbations of the Ricci tensor, Ricci scalar and stress-energy tensor from the background values. The Ricci tensor perturbation can be written as

$$\delta R_{\mu\nu} = \frac{1}{2} \left( h^{\alpha}_{\mu,\alpha} + h^{\alpha}_{\nu,\alpha} - h^{\alpha}_{\alpha,\mu} - h^{\alpha}_{\nu,\alpha} \right).$$

(4.24)
and the perturbation of the Ricci scalar is then

$$\delta R = h^{\alpha\beta,\alpha\beta} - h^{\alpha,\alpha\beta\beta} .$$  \hspace{1cm} (4.25)$$

The perturbation of the stress-energy tensor is more complex and will in general depend on both the perturbation of the metric and the perturbation of the matter-energy content of the spacetime.

From here on we shall assume the spatial section of the metric to be flat ($K = 0$) in order to simplify the equations. We will also use the conformal-Newtonian gauge which simplifies the discussion further. Then the scalar perturbations take the form

$$h_{\mu\nu} = \begin{pmatrix} 2\varphi & 0 \\ 0 & 2\psi \delta_{ij} \end{pmatrix} .$$  \hspace{1cm} (4.26)$$

The Einstein field equations for the perturbation (4.26) are

$$\kappa \delta T_{0}^{0} = \frac{2}{a^2} \left( -3\mathcal{H} [\mathcal{H} \varphi + \psi'] + \nabla^2 \psi \right)$$

$$\kappa \delta T_{i}^{0} = \frac{2}{a^2} (\mathcal{H} + \psi'),$$

$$\kappa \delta T_{j}^{i} = -\frac{2}{a^2} \left( \left(2\mathcal{H}' + \mathcal{H}^2\right)\varphi + \mathcal{H}\varphi' + \psi'' + 2\mathcal{H}\psi' + \frac{1}{2} \nabla^2 D \right) \delta_{ij} - \frac{1}{2} D_{ij},$$

where $\mathcal{H} = a'/a$, the prime denotes differentiation with respect to the conformal time and

$$D = \varphi - \psi .$$  \hspace{1cm} (4.28)$$

4.1.3 Scalar field perturbations

Next we will need the perturbations of the stress-energy tensor. We shall examine the case of a single scalar field for which the stress-energy tensor is

$$T^{\mu\nu} = \phi^{\mu} \phi_{\nu} - \left[ \frac{1}{2} \phi_{,\alpha} \phi_{,\alpha} + \frac{1}{2} m^2 \phi^2 \right] .$$  \hspace{1cm} (4.29)$$

Consider then linear perturbations of the scalar field. The field can then be expanded around the background field $\tilde{\phi}(t)$ as

$$\phi(t, \vec{x}) = \tilde{\phi}(t) + \delta\phi(t, \vec{x}) .$$  \hspace{1cm} (4.30)$$

The first order perturbation of the stress-energy tensor is then

$$\delta T_{0}^{0} = \frac{1}{a^2} \left( -\tilde{\phi}'^2 \varphi + \tilde{\phi}' \delta \phi' - m^2 a^2 \phi \delta \phi \right)$$

$$\delta T_{i}^{0} = \frac{1}{a^2} \tilde{\phi}' \delta \phi_{,i}$$

$$\delta T_{j}^{i} = \left( \tilde{\phi}'^2 \varphi - \tilde{\phi}' \delta \phi' - m^2 a^2 \phi \delta \phi \right) \delta_{ij} .$$  \hspace{1cm} (4.33)$$
Because $\delta T_{ij} \propto \delta_{ij}$, we may set $\psi = \varphi$ and the Einstein equations take the form

$$\nabla^2 \psi - 3H\psi' - 2H^2\psi + H'\psi = \frac{\kappa}{2} \left( \ddot{\phi} \delta \phi' - m^2 a^2 \phi \delta \phi \right)$$

$$\psi' + H\psi = \frac{\kappa}{2} \ddot{\phi} \delta \phi$$

$$\psi'' + 3H\psi' + (H' + 2H^2) \psi = \frac{\kappa}{2} \left( \ddot{\phi} \delta \phi' + m^2 a^2 \phi \delta \phi \right),$$

(4.34)

where we have used the background relation

$$\frac{\kappa}{2} \ddot{\phi}^2 = H^2 - H'. $$

(4.35)

Only two of the equations (4.34) are independent.

From (4.34) we can derive a second order partial differential equation for the scalar perturbation $\psi$

$$\psi'' + 2 \left( \frac{a}{\phi'} \right)' \left( \frac{a}{\phi'} \right)^{-1} \psi' - \nabla^2 \psi + 2 \ddot{\phi} \left( \frac{H}{\phi'} \right) \psi = 0 .$$

(4.36)

By introducing a new variable

$$u = \frac{a}{\phi} \psi$$

(4.37)

and using the equations for the background FRW-model (4.22) given in chapter 6 of [12], equation (4.36) can be reduced to

$$u'' - \nabla^2 u - \left( \frac{\theta''}{\theta} \right) u = 0 ,$$

(4.38)

where

$$\theta = \frac{H}{a\phi'} .$$

(4.39)

Let’s then consider plane wave solutions

$$\psi, \delta \phi, u \propto e^{ikx}$$

(4.40)

in the asymptotic limit. For short wavelength perturbations $k \gg \theta'/\theta$ so the third term in (4.38) can be neglected and we get

$$u \propto e^{\pm ik\eta} .$$

(4.41)

For long wavelength perturbations the second term in (4.38) is much smaller than the third term and we have

$$u \propto (\ddot{\phi})^{-1} \left( \frac{1}{a} \int_{\eta} d\eta a^2(\eta) \right)' .$$

(4.42)

The expressions for $\psi$ and $\delta \phi$ can be easily obtained from (4.37) and (4.34).
4.2 Classical Auxiliary Fields and Vacuum Energy

4.2.1 Simple Examples of Vacuum Energy

Trace Anomaly in Minkowski space

We shall now examine the anomalous stress-energy tensor in flat spacetime. Then $G_{\mu\nu}$ in the second term of (3.65) takes the form

\[
G_{\mu\nu} = -\frac{2}{3} \partial_\mu \partial_\nu \Box \varphi - 2 [ (\partial_\mu \varphi) \partial_\nu + (\partial_\nu \varphi) \partial_\mu ] \Box \varphi + 2 (\partial_\mu \partial_\nu \varphi) \Box \varphi + \frac{1}{6} \eta_{\mu\nu} \left\{ -3 (\Box \varphi)^2 + \Box [ (\partial_\alpha \varphi)(\partial^\alpha \varphi) ] \right\}. \tag{4.43}
\]

The equations of motion for the auxiliary fields reduce in the flat space into the trivial form

\[
\Delta_4 \varphi = \Box^2 \varphi = 0, \quad \Delta_4 \psi = \Box^2 \psi = 0. \tag{4.44}
\]

One particular solution to (4.44) is then

\[
\varphi = a \eta_{\alpha\beta} x^\alpha x^\beta, \quad \psi = 0, \tag{4.45}
\]

which leads to the anomalous stress-energy tensor

\[
T_{\mu\nu}^{\text{anom}} = \frac{16}{3} b' a^2 \eta_{\mu\nu}. \tag{4.46}
\]

This is exactly the form of the cosmological term in the EFE. The constant $a$ and therefore the strength of the "cosmological constant" depends on the exact boundary conditions of the auxiliary field $\varphi$. It is noteworthy that the trace anomaly can lead to a cosmological term in Minkowski space where such term is not naturally present.

Starobinsky’s inflation

Another simple example of vacuum energy is the Starobinsky’s inflation model first derived in [17]. Let’s begin by considering a conformally flat metric $g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}$. In this metric the equations for the auxiliary fields have the simple form

\[
\Box^2 \phi = 2 \Box^2 \sigma, \quad \Box^2 \psi = 0. \tag{4.47}
\]

The solution to these equations is

\[
\phi = 2 \sigma, \quad \psi = 0, \tag{4.48}
\]

which corresponds to the Bunch-Davies vacuum state first derived in [6].
When the solution (4.48) is substituted into the equation for the anomalous stress-energy tensor (3.65), and the resulting stress-energy tensor is used in the flat FRW-model with conformal time, we obtain the third order differential equation for the Hubble constant

\[ \dddot{H} + 7\dot{H}H + 22\dot{H}^2 + 4\dot{H}^2 + 6H^4 - \frac{3}{32\pi^2\kappa b'} \left( H^2 + \dot{H} \right) = 0. \]  

(4.49)

In (4.49) we have converted back into the coordinate time \( t \) and used the identity \( H = \dot{\sigma} \). The constant solution for \( H \) is easily found and is

\[ H = \pm \frac{1}{4\pi\sqrt{2}\kappa b'}, \quad a(t) = a_0 e^{Ht}, \]  

(4.50)

where the positive sign corresponds to inflation. This solution was first discovered by Starobinsky in [17].

### 4.2.2 Classical Degrees of Freedom

Now that we have seen some examples of vacuum energy generated by the trace anomaly we shall consider more thoroughly the effects arising from the anomalous action (3.65) in de Sitter space. The metric of de Sitter spacetime can be represented in the form

\[ g_{\mu\nu} = \text{diag}(1, -a^2(t), -a^2(t), -a^2(t)), \]  

(4.51)

where \( a(t) \) is

\[ a(t) \equiv e^{Ht}. \]  

(4.52)

This particular choice of coordinates coincidences with the flat Friedman-Robertson-Walker metric and they are hence called flat de Sitter coordinates. In these coordinates Riemann tensor, Ricci tensor and Ricci scalar are

\[ R_{\mu\nu\rho\sigma} = H^2 (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \]  

(4.53)

\[ R_{\mu\nu} = 3H^2 g_{\mu\nu} \]  

(4.54)

\[ R = 12H^2. \]  

(4.55)

By substituting these into the Einstein field equations with \( T_{\mu\nu} = 0 \) we see that

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = (3H^2 - 6H^2 + \Lambda)g_{\mu\nu} = 0, \]  

(4.56)

which gives the Hubble constant of de Sitter space \( H^2 = \Lambda/3 \). This also implies that de Sitter space is a vacuum solution of EFE and represents an universe dominated by vacuum energy.

Let’s now make the coordinate transformation into conformal time

\[ \tau = \frac{1}{H} e^{Ht}. \]  

(4.57)
The metric becomes then conformally flat
\[ g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu} = \frac{1}{H^2 + 2} \eta_{\mu\nu} \quad , \tag{4.58} \]
and the Minkowski vacuum state is represented by the auxiliary fields
\[ \varphi = 2Ht = \frac{1}{2} \ln \left( 36H^4 R^{\alpha\beta} R_{\alpha\beta} \right) \tag{4.59} \]
\[ \psi = 0 \quad . \tag{4.60} \]
Now it is clear that \( F_{\mu\nu} = 0 \) and the first term of the anomalous stress-energy tensor (3.65) will be zero. On the other hand \( G_{\mu\nu} \) can be expressed completely in terms of Ricci tensor and Ricci scalar
\[ G_{\mu\nu} = \frac{2}{9} \nabla_\mu \nabla_\nu R + 2R_{\mu}^{\phantom{\mu}\alpha} R_{\nu\alpha} - \frac{14}{9} RR_{\mu\nu} + g_{\mu\nu} \left( -\frac{2}{9} \Box R - R_{\alpha\beta} R^{\alpha\beta} + \frac{5}{9} R^2 \right) . \tag{4.61} \]
This result is easily translated back to the flat coordinates as we already know \( R_{\mu\nu} \) and \( R \) in these coordinates. By substituting (4.54) and (4.55) in to (4.61) we find that
\[ G_{\mu\nu} = 6H^4 g_{\mu\nu} \quad . \tag{4.62} \]
The anomalous energy tensor is then simply
\[ T_{\mu\nu}^{\text{anom}} = 6b'H^4 g_{\mu\nu} \quad , \tag{4.63} \]
which has exactly the form of the cosmological term in EFE, in other word the anomalous stress-energy tensor corresponds to vacuum energy.
Consider then EFE with the anomalous energy tensor
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}^{\text{anom}} . \tag{4.64} \]
By taking the trace we find that
\[ -12H^2 + 4\Lambda = -24\kappa b'H^4 \quad , \tag{4.65} \]
from which we obtain the square of the Hubble constant
\[ H^2 = -\frac{\sqrt{9 - 24\kappa b' \Lambda} - 3}{12b' \kappa} = \frac{\Lambda}{3} \left( 1 + \frac{2\kappa b'}{3} \Lambda \right) + \mathcal{O}(\Lambda^2) \quad , \tag{4.66} \]
expanded around the classical de Sitter solution.
The correction given by the trace anomaly is quite small and it can’t explain the current observations. This can be seen more explicitly by considering a model where the trace anomaly generates a cosmological constant. By setting \( \Lambda = 0 \) in (4.64) we get
\[ -12H^2 = -24\kappa b'H^4 \quad , \tag{4.67} \]
which gives the square of the Hubble constant

\[ H^2 = \frac{1}{2\kappa b'} = \frac{\Lambda_{\text{anom}}}{3}. \]  

(4.68)

The anomalous cosmological constant is then

\[ \Lambda_{\text{anom}} = \frac{2}{3\kappa b'} = \frac{1}{540(4\pi)^2\kappa} \left( N_S + \frac{11}{2} N_F + 62 N_V \right)^{-1}. \]  

(4.69)

The dimensionless value obtained by multiplying \( \Lambda_{\text{anom}} \) with the square of the Planck length is then

\[ \lambda_{\text{anom}} = 4.66... \times 10^{-7} \times \left( N_S + \frac{11}{2} N_F + 62 N_V \right)^{-1}, \]  

(4.70)

which is obviously in contradiction with the experiments, but still better than the ultraviolet estimates discussed in the introduction. Clearly the classical auxiliary fields can’t produce the observed cosmological constant in de Sitter space, but we seem to be on the right track. Next logical step is to consider small perturbations around de Sitter metric, which might lead to a larger effect.

### 4.2.3 Linear Perturbations in de Sitter space

After the failure in the previous section we shall now consider the linear perturbations of the metric, which might amplify the effects of the trace anomaly. In our treatment we follow the one given by Anderson, Molina-Paris and Mottola in [1]. Let’s begin by considering the linear variation of EFE in de Sitter space

\[ \delta R_{\mu \nu} - \frac{1}{2} \delta R g_{\mu \nu} = -\kappa \delta T_{\mu \nu}. \]  

(4.71)

The variation of the Ricci scalar \( \delta R \) will lead only to solutions that depend on the Planck scale and hence we will require \( \delta R = 0 \). By expanding the stress-energy tensor on the left-hand side and bringing the purely geometrical terms onto the right-hand side we get

\[ \delta R_{\mu \nu} - \frac{1}{2} \delta R g_{\mu \nu} = \kappa A_{\mu \nu} + \kappa \alpha \delta A_{\mu \nu} + \kappa \beta B_{\mu \nu} = -\kappa b' \delta F_{\mu \nu} - \kappa b' \delta G_{\mu \nu}, \]  

(4.72)

where \( \delta A_{\mu \nu} \) and \( \delta B_{\mu \nu} \) are the variations of the counterterms in the Lagrangian (2.78). These variations are gauge independent and can be expressed in terms of the variation of the Ricci tensor as

\[ \delta A_{\mu \nu} = 2 \left( -\Box + \frac{R}{3} \right) \delta R_{\mu \nu}, \]  

(4.73)

\[ \delta B_{\mu \nu} = -2 R \delta R_{\mu \nu}. \]  

(4.74)

As we saw in the section 4.1.3 in de Sitter space there is only one scalar degree of freedom. Hence we shall consider only the time-time component of
the linear perturbation equations. The variations $\delta F^{\mu\nu}$ and $\delta G^{\mu\nu}$ are most easily calculated in the gauge

\begin{align*}
h^{ij}g_{ij} &= h^{i\alpha} + h_{00} = 0 \\
\nabla_i h^{0i} &= \frac{1}{2} \partial_0 h_{00}
\end{align*}

in which their time-time components are simply

\begin{align*}
\delta F_{00} &= 4Ht \left( -\Box + 4H^2 \right) \delta R_{00} \\
\delta G_{00} &= -\frac{20H^2}{3} \delta R_{00} .
\end{align*}

By using the variations (4.73), (4.74) and (4.77), (4.78) in equation (4.72) the linear response equation takes the form

\begin{align*}
\delta R_{00} + 2\kappa \alpha \left( -\Box + \frac{R}{3} \right) \delta R_{00} - 2\kappa \beta R \delta R_{00} \\
= -\frac{20kb'H^2}{3} \delta R_{00} - 4\kappa bHt \left( -\Box + 4H^2 \right) \delta R_{00} .
\end{align*}

Now we define the dimensionless variable $q$ as

\begin{equation}
q = -\frac{2a^2}{H^2} \delta R_{00} .
\end{equation}

Then by multiplying (4.79) by $-2a^2/H^2$ and taking the Fourier transform over the spatial variables we find

\begin{equation}
\left( 1 - 2\kappa \beta R + \frac{20kb'H^2}{3} \right) q = -4\kappa bH^3 t Dq - 2\kappa \alpha H^2 Dq ,
\end{equation}

where the differential operator $D$ is defined to be

\begin{equation}
Dq = \frac{1}{H^2} \left( \frac{d^2}{dt^2} + 3H \frac{d}{dt} + 2H^2 + k^2 e^{-2Ht} \right) .
\end{equation}

Consider now the quantities on the left-hand side of (4.81). The last two terms in the brackets are small compared to unity and can hence be ignored leaving us with

\begin{equation}
q = -4\kappa bH^3 t Dq - 2\kappa \alpha H^2 Dq .
\end{equation}

The second term in (4.83) leads to contributions that depend on the Planck scale, which suggests that we should discard it as its contributions are outside the scope of the semiclassical theory. However, the second term is actually comparable to the first one, as shown in chapter VII of [1], and hence it can not be discarded. The only possible semiclassical solution is then the trivial solution $q = 0$, which doesn’t give any new corrections to the cosmological constant.
4.3 Dynamical Effects of Quantized Auxiliary Fields

4.3.1 Anomalous Scaling Dimensions

Because the classical auxiliary fields $\varphi$ and $\psi$ clearly can’t produce the cosmological constant of observations, we shall now quantize these fields and explore the dynamical effects of varying the quantum state. At first we shall consider the scaling of the cosmological constant at large distances as is done by Antoniadis, Mazur and Mottola in [2].

Let’s first consider the classical scaling of $\Lambda$ and $\kappa$ in the Einstein-Hilbert action. In a conformally flat space the action is

$$\frac{1}{\kappa} \int d^4x \left[ 3e^{2\sigma}(\partial_\alpha \sigma)(\partial^\alpha \sigma) - \Lambda e^{4\sigma} \right]. \quad (4.84)$$

The scaling of a term in the Lagrangian is determined by its conformal weight. A term $T_p$ that is proportional to $T_p \propto e^{(4-p)\sigma}$ has a conformal weight of $p$ and a scaling dimension of $\beta_p = 4 - p$, scales like

$$T_p \sim V^{-\frac{4-p}{4}} \quad (4.86)$$

as a function of the four-volume $V$.

The cosmological term is multiplied by $e^{4\sigma}$ and has hence a conformal weight of zero. The quantity $\Lambda/\kappa$ must then be proportional to the inverse of the four-volume

$$\frac{\Lambda}{\kappa} \sim V^{-1}. \quad (4.87)$$

The Einstein curvature term is proportional to $e^{2\sigma}$, has a conformal weight of two and hence $\kappa^{-1}$ is inversely proportional to the square root of the four-volume which yields for $\kappa$ the relation

$$\kappa \sim V^{\frac{1}{2}}. \quad (4.88)$$

In order to make sense of these arbitrary scaling relations we must use some dimensionless quantity for which $\Lambda$ is measured in the units of $\kappa$. A suitable quantity is the dimensionless cosmological constant

$$\lambda = L_{\text{pl}}^2 \Lambda = \frac{\kappa \Lambda}{8\pi}, \quad (4.89)$$

which has the classical scaling dimension

$$\lambda \sim \kappa \Lambda \sim 1. \quad (4.90)$$
In other words the cosmological constant of the classical theory is independent of the scale, just as one would expect.

When we take the trace anomaly into account, the classical scaling dimensions (4.87), (4.88) and (4.89) will obtain anomalous contributions from the fluctuations of the conformal factor \( \sigma \). These fluctuations can be obtained by analyzing the Wess-Zumino action for \( \sigma \) which in conformally flat spacetime takes the form

\[
\Gamma_{\text{WZ}} = 2b' \int d^4x (\Box \sigma)^2 .
\]  

This action is that of a massless, free scalar field with kinetic term of fourth order in derivatives. It is remarkable that all the complicated self-interactions of \( \sigma \) in the Einstein-Hilbert action (4.84) can be treated with the fourth order kinetic term of the WZ-action.

It can be shown ([4], [2]) that all the interaction terms are renormalizable and their anomalous scaling dimensions can be calculated in a closed form. In general the anomalous scaling dimension for a local quantum operator \( O_p \) with non-negative conformal weight \( p \) is given by

\[
\beta_p = 4 - p + \frac{\beta_p^2}{2Q^2} ,
\]  

where \( Q^2 \) depends only on the trace anomaly coefficient \( b' \), that is, the number of the quantum fields

\[
Q^2 \equiv -32\pi^2 b' = \frac{1}{180} \left( N_S + \frac{11}{2} N_F + 62 N_V \right) .
\]  

In order to obtain the classical scaling dimension \( \beta_p^{\text{cl}} = 4 - p \) at the limit \( Q^2 \to \infty \) we must choose the solution

\[
\beta_p = Q^2 \left[ 1 - \sqrt{1 - \frac{8 - 2p}{Q^2}} \right] ,
\]  

which holds for \( Q^2 \geq 8 - 2p \) for all \( p \). Hence we must always have \( Q^2 \geq 8 \). An operator \( O_p \) with a conformal weight \( p \) will then scale with the four-volume as

\[
O_p \sim V^{-\beta_p^{\text{cl}}} .
\]  

The quantities \( \Lambda/\kappa \) and \( \kappa \) with conformal weights of zero and two will then have the anomalous scaling dimensions

\[
\beta_0 = Q^2 \left[ 1 - \sqrt{1 - \frac{8}{Q^2}} \right] \quad \text{and} \quad \beta_2 = Q^2 \left[ 1 - \sqrt{1 - \frac{4}{Q^2}} \right] ,
\]  

and they will scale with the four-volume as

\[
\frac{\Lambda}{\kappa} \sim V^{-1} \quad \text{and} \quad \kappa \sim V^{\beta_2 - \beta_0} \equiv V^\delta .
\]
The dimensionless cosmological constant will then scale as
\[ \lambda \sim \kappa \Lambda \sim V^{2\delta - 1} , \] (4.98)
where the exponent is
\[ 2\delta - 1 \equiv 2\frac{\beta_2}{\beta_0} - 1 = \frac{\sqrt{1 - \frac{8}{Q^2}} - \sqrt{1 - \frac{4}{Q^2}}}{1 + \sqrt{1 - \frac{4}{Q^2}}} \leq 0 \] , (4.99)
for \( Q^2 \geq 8 \).

Because the exponent \( 2\delta - 1 \) is negative, the dimensionless cosmological constant will decrease at large distances. Thus the quantum fluctuations of the conformal factor \( \sigma \) effectively screen the cosmological constant at large distances. In the limit of infinite distance \( V \to 0 \) the cosmological constant flows to zero \( \lambda \to 0 \).

One can also compare the running of \( \kappa \) and \( \Lambda \) to some mass-scale, for example to a mass of a fermion field \( m \), which will scale with the four-volume as
\[ m \sim V^{-\frac{\beta_3}{\beta_0}} . \] (4.100)
Dimensionless quantities formed with \( m \) are then
\[ \kappa m^2 \sim V^{\frac{\beta_2 - 2\beta_3}{\beta_0}} \] (4.101)
\[ \frac{\Lambda}{\kappa m^4} \sim V^{-1 + 4\frac{\beta_3}{\beta_0}} . \] (4.102)

Since
\[ -1 + 4\frac{\beta_3}{\beta_0} < 0 \] (4.103)
for \( Q^2 > 8 \), the cosmological constant decreases at large distances. On the other hand
\[ \frac{\beta_2 - 2\beta_3}{\beta_0} > 0 \] (4.104)
for \( Q^2 > 8 \), and hence the Newtonian constant increases at large distances, which is clearly in contradiction with the current observations. How this result should be treated is not known.

### 4.3.2 Linear Backreaction

After the considerations of the previous section we shall now study the linear response in gravity by examining small perturbations of the metric and the stress-energy tensor expectation value, as Mottola has proposed in [10]. At
first we need to expand the semiclassical Einstein field equations around de Sitter space for which the linear variation is

$$\delta \left\{ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} \right\} = -\kappa \delta \langle T^\mu_{\ \nu} \rangle \quad . \quad (4.105)$$

The left-hand side of (4.105) is purely geometrical, while the right-hand side has two kind of terms. The first are those obtained in 4.2.3. These terms are proportional to the metric variation and will depend on the Planck scale, at which the semi-classical theory is assumed to break down. Hence they can not be trusted and should be discarded.

The second kind of term on the right hand side of (4.105) is obtained by varying the underlying quantum state of the fields independently of the metric. The quantum state depends on the boundary conditions on the cosmological horizon scale, and its variations will lead into terms that are in the region of the semi-classical theory. The anomaly action (3.64) will lead into additional state dependent contributions from the variation of $\delta \langle T^\text{anom}_{\mu \nu} \rangle$, which depends on the quantum state of the auxiliary fields.

When we take into account the variation of the auxiliary fields, the perturbation equation (4.79) is modified to

$$\delta R^0_0 + 2\kappa \alpha \left( -\Box + \frac{R}{3} \right) \delta R^0_0 - 2\kappa \beta R \delta R^0_0$$

$$= -\frac{20\kappa b' \beta H^2}{3} \delta R^0_0 - 4\kappa b H t \left( -\Box + 4H^2 \right) \delta R^0_0$$

$$+ \frac{2\kappa b' \beta H^2}{3a^2} \hat{\nabla}^2 u - \frac{2\kappa b H^2}{3a^2} \hat{\nabla}^2 v \quad , \quad (4.106)$$

where $u$ and $v$ are the gauge invariant variations of the auxiliary fields $\phi$ and $\psi$

$$u = H^{-2} \left( \frac{d^2}{dt^2} + H \frac{d}{dt} - \frac{\hat{\nabla}^2}{a^2} \right) \delta \varphi - 2h_{00} \quad (4.107)$$

$$v = H^{-2} \left( \frac{d^2}{dt^2} + H \frac{d}{dt} - \frac{\hat{\nabla}^2}{a^2} \right) \delta \psi \quad . \quad (4.108)$$

By using the dimensionless quantity $q$ from (4.80) and Fourier transforming, the perturbation equation (4.106) can be converted into

$$\left( 1 - 2\kappa \beta R + \frac{20\kappa b' \beta H^2}{3} \right) q = -4\kappa b H^3 t \mathcal{D} q - 2\kappa \alpha H^2 \mathcal{D} q + \frac{2\kappa b'}{3a^2} k^2 u - \frac{2\kappa b}{3a^2} k^2 v \quad , \quad (4.109)$$

where $\mathcal{D} q$ is defined in (4.82). The equations for $u$ and $v$ in Fourier space are
given in chapter VI of [1] and are simply
\[
\left( \frac{d^2}{dt^2} + 5H \frac{d}{dt} + 6H^2 + \frac{k^2}{a^2} \right) u = 0 \tag{4.110}
\]
\[
\left( \frac{d^2}{dt^2} + 5H \frac{d}{dt} + 6H^2 + \frac{k^2}{a^2} \right) v = 0 . \tag{4.111}
\]

The general solutions for \( u \) and \( v \) are the linear combinations of the mode-solutions
\[
u_\pm = v_\pm = \frac{1}{H^2 a^2} e^{\pm \frac{2\eta k}{3a^2} + ik_2 \cdot x} = \eta^2 e^{\mp ik_2 \cdot x} . \tag{4.112}
\]

These modes are independent of the Planck scale and are therefore genuine low energy modes of the semiclassical theory. When these solutions are substituted into (4.109), \( Dq \) terms can be ignored. When the two last terms on the left-hand side are ignored as well we obtain
\[
q \approx \frac{2\kappa b'}{3a^2} k^2 \eta^2 e^{ik_2 \cdot x} - \frac{2\kappa b'}{3a^2} k^2 \eta^2 e^{ik_2 \cdot x} . \tag{4.113}
\]

The associated stress-energy tensor perturbation for the time-time component is
\[
\delta \langle T^{00} \rangle = \frac{H^2}{2\kappa a^2} q = \frac{b' H^2}{3a^2} k^2 \eta^2 e^{ik_2 \cdot x} - \frac{b H^2}{3a^2} k^2 \eta^2 e^{ik_2 \cdot x} . \tag{4.114}
\]

Other components of the stress-energy tensor perturbation can be found by requiring the covariant conservation
\[
\nabla_\alpha \delta \langle T^\mu_\alpha \rangle = 0 , \tag{4.115}
\]
and the trace free condition
\[
\delta \langle T^\alpha_\alpha \rangle = 0 , \tag{4.116}
\]
which is a result of the condition \( \delta R = 0 \). The components are then
\[
\delta \langle T^{00} \rangle = b' H^2 k^2 \eta^2 e^{ik_2 \cdot x} - \frac{b' H^2}{a^2} \nabla^2 \eta^2 e^{ik_2 \cdot x} - \frac{b H^2}{a^2} \nabla^2 \eta^2 e^{ik_2 \cdot x} . \tag{4.117}
\]
\[
\delta \langle T^{ij} \rangle = \pm b' H^2 k^2 \eta^2 e^{ik_2 \cdot x} - \frac{b' H^2}{a^2} \nabla^2 \eta^2 e^{ik_2 \cdot x} - \frac{b H^2}{a^2} \nabla^2 \eta^2 e^{ik_2 \cdot x} ,
\]
where we have left out the \( b \) term, for which the results are similar.

It should be noted that the result (4.117) would be obtained even if the anomalous action alone was used to generate the linear response. Thus the anomalous action is needed to describe the physical infrared fluctuations of the semiclassical quantum gravity on cosmological scale. In the limit of flat space or weak coupling \( \kappa H^2 \to 0 \) the infrared modes \( u \) and \( v \) decouple from the metric perturbation at linear order and we are left with the classical equations.
4.3.3 Cosmological Horizon Modes

In order to understand the physics behind the modes $u$ and $v$ we shall now examine the stress-energy tensor they produce. First we should note that the stress-energy tensor (4.117) is traceless, and cannot hence be deduced straight from the trace anomaly. Instead we must use the quantized auxiliary fields and let them fluctuate around the vacuum solution of the fields. The resulting modes are thus part of the first-loop quantum effects.

One interpretation for (4.117) can be found by averaging over the directions of $\vec{k}$. The resulting stress-energy tensor is then spatially homogeneous and isotropic, with equation of state $p = \rho/3$. Thus it describes incoherent massless radiation in the flat FRW coordinates.

A more enlightening interpretation is found by considering (4.117) in static coordinates of de Sitter space, in which the line-element is

$$ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .$$

The coordinate transformation from the flat coordinates to the static ones is

$$r = |\vec{x}| e^{Ht}$$

$$\tau = t - \frac{1}{2H} \ln \left( 1 - H^2 |\vec{x}|^2 e^{2Ht} \right) .$$

In these coordinates the solution for $u$ and $v$ becomes

$$u = v = \frac{4}{1 - H^2 r^2} ,$$

which is time-independent and diverges at the cosmological horizon $r = 1/H^2$.

The stress-energy tensor corresponding to the solution (4.121) in the static coordinates is then

$$\delta \langle T^\mu_\nu \rangle = \frac{6H^4}{(1 - H^2 r^2)^2} \delta^\mu_\nu$$

$$\delta \langle T^\mu_i \rangle = 0$$

$$\delta \langle T^i_j \rangle = -\frac{2H^4}{(1 - H^2 r^2)^2} \delta^i_j .$$

The complete calculation is given in [10] by Mottola. Again we find that the stress-energy tensor corresponds to a perfect fluid with equation of state $p = \rho/3$, but now in the static coordinates. It is also remarkable that (4.122) diverges quartically on the cosmological horizon $r = 1/H^2$. This suggests that the horizon of de Sitter space might be more dynamic than it has been thought before. Because the effects of the modes $u$ and $v$ build up on the cosmological horizon of de Sitter space, these modes may be called cosmological horizon modes.

The form of the stress-energy tensor (4.122) is in fact the form of a finite temperature fluctuation away from the Hawking-de Sitter temperature

$$T_H = \frac{H}{2\pi} .$$

(4.123)
of the vacuum state. This indicates that the stress-energy tensor is due a change of boundary conditions of the quantum fields on the horizon. The divergence on the horizon then signals the breakdown of these boundary conditions, and suggests that the horizon is a physical entity rather than a purely mathematical boundary as has been thought before. However, the linear perturbation theory breaks down near the horizon where non-linear backreaction effects may be expected, and hence no decisive conclusions can be made of the nature of the horizon while regarding only linear perturbations.

The stress-energy fluctuation (4.122) is isotropic, but spatially inhomogeneous. Thus it breaks the de Sitter isometry group $O(4,1)$ preserving the $O(3)$ subgroup. The origin $r = 0$ is clearly arbitrary, and the particular $O(3)$ subgroup is chosen at random. In other words the de Sitter group is spontaneously broken by any random fluctuation of the temperature, and it turns out that the perturbation will in fact destabilize the global de Sitter space. This instability is related to the non-zero cosmological constant and provides a possible way to distinguish different values of the vacuum energy. In absence of any boundary conditions $\Lambda = 0$ would then be automatically selected, which implies that the observed small cosmological constant is a result of physical border conditions on the horizon of de Sitter space.
Conclusions

As we have seen, the semiclassical quantum gravitation provides us with means to study the quantum effects on large distance scales. We have seen that the semiclassical theory contains a trace anomaly which can't be removed without breaking the conformal symmetry or using non-local action. The trace anomaly itself was represented by a non-local anomalous action, which depends on global border conditions. In order to render the anomalous action local we introduced two new classical fields, in which the non-local border conditions were contained. These new fields then added two new degrees of freedom into the Lagrangian.

The addition of border condition sensitive auxiliary fields didn't at first seem to have much of an impact to the cosmological constant. However, when the auxiliary fields itself were quantized and allowed to fluctuate freely, the cosmological constant acquired an anomalous scaling dimension. This scaling dimension implied, that the cosmological constant is driven into zero at large distances, which would certainly explain the small value obtained from the observations. Unfortunately the quantization of the auxiliary fields also drives the Newtonian constant into infinity at large distances, which seems to contradict the observations.

On a closer inspection we found cosmological horizon scale wave-modes for the auxiliary fields. When averaged over all possible directions these wave-modes contributed to the stress-energy tensor as thermal radiation. After this remark we made a coordinate transformation into static coordinates where the general solution for the wave-modes became time-independent. The wave-modes and the associated stress-energy tensor were found to be divergent at the horizon of de Sitter space, and this divergent behavior was traced back to border conditions on the horizon. Closer inspection revealed that the fluctuations of the auxiliary fields rendered de Sitter space unstable, and it was deduced that in absence of any border conditions the cosmological constant would be zero. Thus the small observed cosmological constant was deduced to be a result of physical border conditions on the horizon.

As a whole the approach we have described seems to work. The problems lie mainly on the semiclassical theory and the interpretation of the anomalous scaling dimensions, as well as the actual significance of the horizon scale wave modes. The validity of the semiclassical approach and the instabilities encountered in 2.3 are concerning, but for now semiclassical theory is all we have and there is no experimental evidence strictly against it. The problem with the anomalous scaling dimensions, the Newtonian constant flowing to in-
finity, might just be an artifact of the calculation methods and may not have any physical significance. Last but not least there is the possibility that the horizon scale wave-modes turn out to be ineffectual in forcing the cosmological constant to the small observed value, but this requires more closer inspection.

On the other hand the trace anomaly could be the solution to the problem of the cosmological constant. It certainly could explain the small value of $\lambda$ without any fine-tuning or additional assumptions. In spite of the concerns noted above the results are promising and the subject certainly deserves more attention. Especially the interaction between the cosmological constant, the horizon scale wave-modes and the horizon of de Sitter space requires more research.
Appendix A

General expressions for the tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ which constitute the anomalous energy tensor

\[ T_{\mu\nu}^{\text{anom}} = b F_{\mu\nu} + b' G_{\mu\nu} \]

\[ F_{\mu\nu} = \]
\[ -2(\nabla_{(\mu})\phi)(\nabla_{\nu})\Box\psi) - 2(\nabla_{(\mu})\psi)(\nabla_{\nu})\Box\phi) - \frac{4}{3} \nabla_\mu \nabla_\nu [(\nabla_\chi\phi)(\nabla^\chi\psi)] \]
\[ + 2 \nabla^\alpha [(\nabla_\alpha\phi)(\nabla_\mu\nabla_\nu\Box\psi) + (\nabla_\alpha\psi)(\nabla_\mu\nabla_\nu\Box\phi)] + \frac{4}{3} R_{\mu\nu}(\nabla_\chi\phi)(\nabla^\chi\psi) \]
\[ - 4 R^\lambda_{(\mu} [(\nabla_\nu)\phi)(\nabla_\lambda\psi) + (\nabla_\nu)\psi)(\nabla_\lambda\phi)] + \frac{4}{3} R(\nabla_{(\mu})\phi)(\nabla_{\nu)})\psi \]
\[ + \frac{1}{3} g_{\mu\nu} \{ -3(\Box\phi)(\Box\psi) + \Box [(\nabla_\chi\phi)(\nabla^\chi\psi)] + 2 (3 R^{\alpha\beta} - R g^{\alpha\beta}) (\nabla_\alpha\phi)(\nabla_\beta\psi) \}
\[ - 4 \nabla_\alpha \nabla_\beta (C_{(\mu}^{\alpha} \nu \beta \phi) - 2 C_{(\mu}^{\alpha} \nu \beta R_{\alpha\beta\phi} - \frac{2}{3} \nabla_\mu \nabla_\nu \Box\psi - 4 C_{(\mu}^{\alpha} \nu \beta \nabla_\alpha \nabla_\beta \psi \]
\[ - 4 R^\alpha_{(\mu}(\nabla_\nu)\nabla_\alpha\psi) + \frac{8}{3} R_{\mu\nu}\Box\psi + \frac{4}{3} R \nabla_\mu \nabla_\nu \psi - \frac{2}{3} (\nabla_{(\rho})\nabla_{\nu)}\psi \]
\[ + \frac{1}{3} g_{\mu\nu} \{ 2 \Box^2 \psi + 6 R^{\alpha\beta}\nabla_\alpha \nabla_\beta \psi - 4 R\Box\psi + (\nabla^\alpha R)(\nabla_\alpha\psi) \} \]

\[ G_{\mu\nu} = \]
\[ -2(\nabla_{(\mu})\phi)(\nabla_{\nu})\phi) + 2 \nabla^\alpha [(\nabla_\alpha\phi)(\nabla_\mu\nabla_\nu\phi)] - \frac{2}{3} \nabla_\mu \nabla_\nu [(\nabla_\chi\phi)(\nabla^\chi\phi)] \]
\[ + \frac{2}{3} R_{\mu\nu}(\nabla_\chi\phi)(\nabla^\chi\phi) - 4 R^\lambda_{(\mu} [(\nabla_\nu)\phi)(\nabla_\lambda\phi)] + \frac{2}{3} R(\nabla_{(\mu})\phi)(\nabla_{\nu)})\phi \]
\[ + \frac{1}{6} g_{\mu\nu} \{ -3(\Box\phi)^2 + \Box [(\nabla_\chi\phi)(\nabla^\chi\phi)] + 2 (3 R^{\alpha\beta} - R g^{\alpha\beta}) (\nabla_\alpha\phi)(\nabla_\beta\phi) \}
\[ - \frac{2}{3} \nabla_\mu \nabla_\nu \Box\phi - 4 C_{(\mu}^{\alpha} \nu \beta \nabla_\alpha \nabla_\beta \phi - 4 R^\alpha_{(\mu}(\nabla_\nu)\nabla_\alpha\phi) + \frac{8}{3} R_{\mu\nu}\Box\phi + \frac{4}{3} R \nabla_\mu \nabla_\nu \phi \]
\[ - \frac{2}{3} (\nabla_{(\mu})\nabla_{\nu})\phi + \frac{1}{3} g_{\mu\nu} \{ 2 \Box^2 \phi + 6 R^{\alpha\beta}\nabla_\alpha \nabla_\beta \phi - 4 R\Box\phi + (\nabla^\alpha R)(\nabla_\alpha\phi) \} \]
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